Institute for Applied Mathematics
SS 2021
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Sheet 6, "Stochastic Analysis"

To be discussed on June 02, 2021

Problem 1 (Max-stable distributions)

Show that the Gumbel, Fréchet and Weibull distribution are max-stable: for all $n \in \mathbb{N}$, there exist $a_n > 0, b_n$, such that $F^n(a_n x + b_n) = F(x) \, \forall x \in \mathbb{R}$.

For all following exercises, let $\{X_i\}_{i \in \mathbb{N}}$ be an i.i.d sequence, where X_1 has distribution F. Define $M_n := \max_{i \leq n} X_i$.

Problem 2 (Examples of extremal limits)

Find out to which domain of attraction the extremal extremal processes M_n of the following distributions belong. Then find suitable sequences $\{a_n\}_{n \in \mathbb{N}} > 0$ and $\{b_n\}_{n \in \mathbb{N}}$, such that $\mathbb{P}[M_n \leq a_n x + b_n]$ converges to a non-degenerate limit.

1. $F(x) = x$, the uniform distribution on $(0, 1)$.
2. $F(x) = 1 - Kx^{-\alpha}$ for $\alpha > 0, K > 0$, the Pareto distribution on $[K^{1/\alpha}, \infty)$.
3. $F(x) = 1 - e^{-x}$, the exponential distribution with rate 1 on $[0, \infty)$.

   Hint: For the exponential distribution, find a suitable non-linear scaling $s(x, n)$ of M_n, then use a Taylor-Expansion to get a_n and b_n.

Problem 3 (Extremal limit of the normal distribution)

Proof that if $F = \Phi$ is the standard normal distribution, then

$$\mathbb{P}[M_n \leq a_n x + b_n] \to \exp(-e^{-x}), \text{ where}$$
$$a_n = (2 \log n)^{-1/2}, \quad b_n = (2 \log n)^{1/2} - \frac{1}{2}(2 \log n)^{-1/2}(\log \log n + \log 4\pi).$$

Hint: You may use without proof that $1 - \Phi(x) \sim \frac{\phi(x)}{x}$ as $x \to \infty$. As a starting point, find a suitable non-linear scaling.

Problem 4 (Distributions with degenerate extremal limit, part 1)

Define $x_F := \sup\{x : F(x) < 1\}$. Assume that $x_F < \infty$ and that $\mathbb{P}[X_1 = x_F] > 0$. Show that for any sequence u_n, such that $\mathbb{P}[M_n \leq u_n] \to \rho$, it must be that $\rho \in \{0, 1\}$.

This statement can be extended to the following Theorem, that gives a precise requirement on the tail of the distribution F. There exists a relatively short proof that requires only Lemma 4.4 from the lecture. It is left as a challenging bonus exercise.

Theorem. For any $\rho \in (0,1)$, there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$ with a degenerate limit $\lim_{n\to\infty} P[M_n \leq u_n] \to \rho$, if and only if

$$\lim_{x\to X_F} \frac{1-F(x)}{1-F(x_-)} = 1,$$

where $F(x_-) := \mathbb{P}[X < x]$, or equivalently if and only if

$$\lim_{x\to X_F} \frac{F(x)-F(x_-)}{1-F(x_-)} = 0.$$

Problem 5 (Distributions with degenerate extremal limit, part 2)
Use the previous statement to conclude that the following distributions do not have a non-degenerate extremal limit:

1) The Poisson distribution with parameter $\lambda > 0$:

$$\mathbb{P}[X = k] = e^{-\lambda}\frac{\lambda^k}{k!} \text{ for } k \in \mathbb{N}_0.$$

2) The Geometric distribution with acceptance probability $p \in (0,1)$:

$$\mathbb{P}[X = k] = (1-p)^{(k-1)} \cdot p \text{ for } k \in \mathbb{N}_0.$$