Institute for Applied Mathematics
SS 2021
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Sheet 5, "Stochastic Analysis"

To be discussed on May 19, 2021

Problem 1 (Laplace functional of a Poisson point process)

Let λ be a finite measure on $\mathbb{R}^d$ and let τ be a Poisson distributed random variable with parameter $\lambda(\mathbb{R}^d)$. Let $(X_i)_{i\in\mathbb{N}}$ be a family of i.i.d. random variables with distribution $\lambda/\lambda(\mathbb{R}^d)$, independent of τ. Define

$$N := \sum_{i=1}^{\tau} \delta_{X_i}.$$

Compute the Laplace functional of N and conclude that it is a Poisson point process with intensity measure λ. Show that for every $f \geq 0$:

$$\Psi_N(f) := \mathbb{E}[e^{-N(f)}] = \exp\Big(- \int_{\mathbb{R}^d} (1 - e^{-f(x)})\, \lambda(dx)\Big).$$

Remark: This equality can easily be extended to complex-valued functions $f : \mathbb{R}^d \mapsto \mathbb{C}$, as long as $\int_{\mathbb{R}^d} |1 - e^{-f(x)}|\lambda(dx) < \infty$. We will need this general version in the following.

Problem 2 (Lévy–Itô decomposition, explicit construction of a Lévy Process)

Let $b \in \mathbb{R}^d$, let $M \in \mathbb{R}^{d\times d}$ be a non-negative definite matrix, and let λ be a measure on $\mathbb{R}^d \setminus \{0\}$ such that $\int (1 \wedge |x|^2)\lambda(dx) < \infty$. For $\theta \in \mathbb{R}^d$, set

$$\Psi(\theta) := i\langle b, \theta\rangle - \frac{1}{2}\langle \theta, M\theta\rangle + \int_{\mathbb{R}^d} \Big(e^{i\langle \theta, x\rangle} - 1 - i\langle \theta, x\rangle \cdot \mathbb{1}_{\{|x|\leq 1\}}\Big)\lambda(dx).$$

We explicitly construct a Lévy process $(X_t)_{t\geq 0}$, such that its characteristic exponent is given by $\mathbb{E}[\exp(i\langle\theta, X_t\rangle)] = \exp(t\cdot\Psi(\theta))$. Thus, X_t is the Lévy process for the triple (b, M, λ).

1) Consider a d-dimensional Brownian motion $(B_t)_{t\geq 0}$ and let $\sqrt{M}$ be a root of M. Define $X_t^{(1)} := \sqrt{M}B_t + bt$. Show that $(X_t^{(1)})_{t\geq 0}$ is a Lévy process and compute its characteristic exponent $\Psi^{(1)}$.

Next, let $\mathscr{P}$ be a Poisson point process on $[0, \infty) \times \mathbb{R}^d$ with intensity measure $dt \times \lambda$, that is independent of $(B_t)_{t\geq 0}$. We split $\mathscr{P} = \mathscr{P}^{(2)} + \mathscr{P}^{(3)}$ into its mass for large and small values of x. Therefore, let $\lambda^{(2)}(dx) := \lambda(dx) \cdot \mathbb{1}_{\{|x|>1\}}$ and $\lambda^{(3)}(dx) := \lambda(dx) \cdot \mathbb{1}_{\{|x|\leq 1\}}$. Let $\mathscr{P}^{(2)}$

and $\mathscr{P}^{(3)}$ be the corresponding Poisson point processes. Note that $\lambda^{(2)}$ is finite, whereas this does not necessarily hold for $\lambda^{(3)}$.

2) Let $X_t^{(2)} := \int_0^t \int_{\mathbb{R}^d} x \mathscr{P}^{(2)}(ds, dx)$. Show that $(X_t^{(2)})_{t\geq 0}$ is a Lévy process and compute its characteristic exponent $\Psi^{(2)}$.

3) Now we deal with the possibly infinitely many jumps induced by $\mathscr{P}^{(3)}$ via an approximation. For $\epsilon > 0$, define

$$X_t^{(3,\epsilon)} := \int_0^t \int_{\mathbb{R}^d} x \cdot \mathbb{1}_{\{\epsilon<|x|\leq 1\}} \mathscr{P}^{(3)}(ds, dx) - t \int_{\mathbb{R}^d} x \cdot \mathbb{1}_{\{\epsilon<|x|\leq 1\}} \lambda(dx).$$

Show that $X_t^{(3,\epsilon)}$ is a Lévy process and compute its characteristic exponent $\Psi^{(3,\epsilon)}$.

4) Proof the following maximal inequality: given a Poisson point process $\mathscr{P}$ with intensity measure $dt \times \lambda$, where λ is finite, then for any function $f \in L^2(\lambda)$:

$$\mathbb{E}\Big(\sup \Big\{ \Big| \int_0^t \int_{\mathbb{R}^d} f(x) \mathscr{P}(ds, dx) - t \int_{\mathbb{R}^d} f(x) \lambda(dx) \Big|^2, 0 \leq t \leq T \Big\} \Big) \leq 4T \int_{\mathbb{R}^d} f(x)^2 \lambda(dx).$$

Hint: Use the explicit construction of a Poisson point process from Problem 1.

5) Let $\epsilon \to 0$ and show that $(X_t^{(3,\epsilon)})_{\epsilon>0}$ forms a Cauchy-sequence on any finite time-inerval $[0, T]$ w.r.t. the norm

$$||Y||_T = \mathbb{E}\big[\sup \big\{ |Y_s|^2 : s \in [0, T] \big\} \big]^{1/2}.$$

You may assume without proof that $\lim_{\epsilon\to 0} X^{(3,\epsilon)} = X^{(3)}$ has càdlàg paths. Show that $X^{(3)}$ is also a Lévy process and compute its characteristic exponent $\Psi^{(3)}$.

6) Check that $X^{(1)}, X^{(2)}$ and $X^{(3)}$ are independent and conclude that their sum is the Lévy process for the triple (b, M, λ).