Institute for Applied Mathematics
SS 2021
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Sheet 4, "Stochastic Analysis"

To be discussed on May 12, 2021

Problem 1 (Scale function: recurrence and exit times)
Suppose that $(X_t)_{0\le t\le \xi}$ is a strong solution of the SDE

$$\begin{aligned} dX_t &= b(X_t)dt + \sigma(X_t)dB_t, \quad t \in [0,\xi), \\ X_0 &= x_0, \end{aligned}$$

for a one-dimensional Brownian motion B_t, an initial value $x_0 \in (0,\infty)$ and continuous b, σ. For simplicity, we fix the left boundary to 0 and define the explosion time ξ as

$$\xi := \sup_{\epsilon,r>0} T_{\epsilon,r}, \quad T_{\epsilon,r} = \inf\{t \ge 0 | X_t \notin (\epsilon, r)\}.$$

The scale-function $s : (0,\infty) \to \mathbb{R}$ of this process is given by

$$s(x) := \int_{x_0}^{x} \exp\Big[- \int_{x_0}^{z} \frac{2b(y)}{\sigma(y)^2} dy \Big] dz.$$

One can easily show that

$$\mathbb{P}[T_\epsilon < T_r] = \frac{s(r) - s(x)}{s(r) - s(\epsilon)}.$$

1) Conclude that:

1. If $s(0) > -\infty$ and $s(\infty) < \infty$, then

$$\mathbb{P}\big[\lim_{t\to\xi} X_t = 0\big] = \frac{s(\infty) - s(x_0)}{s(\infty) - s(0)}.$$

2. If $s(0) > -\infty$ and $s(\infty) = \infty$, then $\lim_{t\to\xi} X_t = 0$ a.s. .
3. If $s(0) = -\infty$ and $s(\infty) < \infty$, then $\lim_{t\to\xi} X_t = \infty$ a.s. .
4. If $s(0) = -\infty$ and $s(\infty) = \infty$, then X_t is recurrent. For any $x_0, y \in (0,\infty)$:

$$\mathbb{P}[X_t = y \text{ for some } t \in [0,\xi)] = 1.$$

2) Let $\alpha \in \mathbb{R}$ and $\sigma > 0$. Let $(S_t)_{0 \le t < \xi}$ be a Geometric Brownian motion, a strong solution of the SDE

$$dS_t = \alpha S_t dt + \sigma S_t dB_t, \quad S_0 = x_0 > 0.$$

Analyze the asymptotic behavior of S_t.

3) Let $\beta \in \mathbb{R}$ and $\sigma > 0$. Let $(F_t)_{0 \le t < \xi}$ be a strong solution of Feller's branching diffusion:

$$dF_t = \beta F_t dt + \sigma \sqrt{F_t} dB_t, \quad F_0 = x_0 > 0.$$

Analyze the asymptotic behavior of F_t.

Remark: The process F_t is the diffusive limit of a discrete Galton-Watson branching process, describing a population in which the number N of offsprings of each individual has the following properties: $\mathbb{E}[N] = 1 + \beta, \mathrm{Var}[N] = \sigma^2$. Hence, $\mathbb{P}[F(\xi) = 0]$ is the probability that this population dies out.

Problem 2 (Local time and mode of continuity of Brownian motion)

Let $(B_t)_{t \ge 0}$ be a one-dimensional standard Brownian motion and let Γ_t be the occupation measure as defined in the lecture. Its density l_t^x is jointly continuous in t and x almost surely. Show that for almost all ω, the path $t \mapsto B_t(\omega)$ is not locally Lipschitz continuous.

Hint: Consider $\Gamma_{t+h}(S_h) - \Gamma_t(S_h)$ for suitable sets S_h.