Institute for Applied Mathematics
SS 2021
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Sheet 3, "Stochastic Analysis"

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Problem 1 (The Feynman-Kac formula)

Let $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ be continuous, such that they grow at most linear, in the sense that $||b(x)||^2 + ||\sigma(x)||^2 \leq K(1 + ||x||^2)$. Under these conditions, the existence of a unique solution of the process $(X_s)_{s\geq t}$ is granted, that under $\mathbb{P}_{t,x}$ solves

$$dX_s = b(X_s)ds + \sigma(X_s)dB_s, \quad X_t = x \in \mathbb{R}^d.$$

Further, it can be shown that these growth conditions let us control $||X||^p$. For every finite time-horizon T and $p \geq 2$, there exists a constant C, such that for all $t \leq s \leq T$:

$$\mathbb{E}_{t,x}\Big[\max_{t\leq\tau\leq s} ||x_\tau||^p\Big] \leq C(1 + ||x||^p) \cdot e^{(s-t)}.$$

Now, fix a finite $T > 0$ and let the functions k, g, f be continuous and non-negative. Assume that $v : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ is of class $C^{(1,2)}([0, T] \times \mathbb{R}^d)$ and solves the Cauchy problem

$$\begin{aligned} -\frac{\partial}{\partial t} v(t,x) &= (\mathscr{L} v)(t,x) - k(t,x) \cdot v(t,x) + g(t,x), \quad x \in \mathbb{R}^d, t \in [0,T), \\ v(T,x) &= f(x), \quad x \in \mathbb{R}^d. \end{aligned}$$

For $a = \sigma\sigma^T$, the differential operator $\mathscr{L}$ is given by

$$\mathscr{L} = \frac{1}{2}\sum_{i,j}^d a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_i^d b_i(x)\frac{\partial}{\partial x_i}.$$

Further, assume that v grows at most polynomially:

$$\max_{0\leq t\leq T} |v(t,x)| \leq K(1 + x^q), \quad \text{for some } q \geq 1.$$

Show that

$$v(t,x) = \mathbb{E}_{t,x}\Big[f(X_T)e^{-\int_t^T k(s,X_s)ds} + \int_t^T g(s,X_s)e^{-\int_t^s k(r,X_r)dr}ds\Big].$$

Hint: Consider

$$Y(s) := e^{-\int_t^s k(\tau,x_\tau)d\tau} \cdot v(s,X_s) + \int_t^s e^{-\int_t^r k(\tau,X_\tau)d\tau} \cdot g(r,X_r)dr.$$

Remark: Not only implies this uniqueness of the solution v, but also allows to compute it via Monte Carlo methods.
Remark: There exist different conditions yielding integrability of the above expressions.

Problem 2 (Harmonicity of the exit-distribution)
Let $D \subset \mathbb{R}^d$ be an open set (that can also be unbounded) and let $f : \partial D \to \mathbb{R}$ be a continuous and bounded function on the boundary of D. Under $\mathbb{P}_x$, let $(B_t)_{t\geq 0}$ be a Brownian motion starting in x and $T := \inf\{t \geq 0 : B_t \notin D\}$. For $x \in D$, define

$$v(x) := \mathbb{E}_x[f(B_T) \cdot \mathbb{1}(T < \infty)].$$

Under the assumption that $v \in C^2(D) \cap C(\bar{D})$, show that for all $x \in D$:

$$\triangle v(x) = 0.$$