Institute for Applied Mathematics
SS 2021
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Sheet 2, "Stochastic Analysis"

To be discussed on April 28, 2021

Problem 1 (Regularity of open sets in $\mathbb{R}^1$)
Let $D \subset \mathbb{R}^1$ be an open and bounded interval with boundary ∂D. Show that ∂D is regular.

Problem 2 (Regularity of C^1-boundaries)
Let $D = \{x \in \mathbb{R}^d : g(x) < 0\}$, where $g \in C^1(\mathbb{R}^d, \mathbb{R})$. Assume that $\nabla g(x) \neq 0$ for all $x \in \partial D$. Show that ∂D is regular.

Problem 3 (Uniqueness of the inhomogeneous heat equation)
Suppose that $g(t,x) : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ is a bounded and continuous function. Assume that $h(t,x) \in C^{1,2}((0,\infty) \times \mathbb{R}^d, \mathbb{R})$ solves

$$\frac{dh}{dt} = \frac{1}{2}\triangle_x h + g(t,x), \tag{1}$$

$$h(0,x) = 0, \text{ and } h \text{ is continuous up to the boundary } \{t = 0\}. \tag{2}$$

For simplicity, assume that $h(t,x)$ is bounded on $[0,T] \times \mathbb{R}^d$ for any finite time-horizon T.

1) Show that, for a finite time-horizon $T > 0$:

$$M_s := h(T-s, B_s) + \int_0^s g(T-r, B_r)\, dr$$

is a martingale on $[0,T)$. Here, under the measure $\mathbb{P}_x$, B_t is a Brownian motion starting in x.

2) Show that $h(x,t) = v(x,t)$, where $v(x,t)$ is defined as

$$v(x,t) := \mathbb{E}_x\Big[\int_0^t g(t-s, B_s)\, ds\Big].$$

3) Now, conversely, assume that $v(t,x)$ as defined before lies in $C^{1,2}((0,\infty) \times \mathbb{R}^d, \mathbb{R})$. Show that v fulfills (1) and (2) (without assuming the existence of $h(t,x)$).

Remark: A solution for $u(0,x) = f_0(x) \neq 0$ can be constructed as a sum of u_1 and u_2, where u_1 solves the homogeneous equation $u_t = \frac{1}{2}\triangle_x u, u(0,x) = f_0(x)$, and u_2 solves the inhomogeneous equation $u_t = \frac{1}{2}\triangle_x u + f, u(0,x) = 0, u(0,x) = 0$.