Institute for Applied Mathematics
SS 2021
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Sheet 10, "Stochastic Analysis"

To be discussed on June 30, 2021

Problem 1 (Boundedness of Lévy processes)

Let $V_{\alpha,c}(t)$ be a Lévy process with Lévy triple $(0, 0, v_{\alpha,c})$, where $v_{\alpha,c}(dx) = c\alpha x^{-\alpha-1}\mathbb{1}_{x>0}dx$ for some $\alpha \in (0, 1), c > 0$. Show that for every $T > 0$ and every $\delta > 0$, there exists a finite constant K, such that

$$\mathbb{P}\Big[\sup_{t\in[0,T]} V_{\alpha,c}(t) \geq K\Big] \leq \delta.$$

Hint: Use two truncation arguments to deal with the heavy tail of $v_{\alpha,c}$ and its mass around the origin.

Now take a sequence $\{X_i\}_{i\in\mathbb{N}}$ of random variables such that $n\mathbb{P}[X_1 > n^{1/\alpha}] \to cx^{-\alpha}$ as $n \to \infty$ and define $S_n(t) := n^{-1/\alpha} \sum_{i=1}^{[nt]} X_i$. Conclude - with the help of Theorem 7.5. - that for every $T > 0$ and every $\epsilon > 0$, there exists a finite constant K, such that for all $n \in \mathbb{N}$:

$$\mathbb{P}[\sup_{t\in[0,T]} S_n(t) \geq K] \leq \epsilon.$$

Problem 2 (Tightness on the space of càdlàg functions)

Let E be a complete metric space. Show that if a sequence of stochastic processes $\{X_n\}_{n\in\mathbb{N}}$ is tight in $D_E[0,\infty)$ equipped with the Skorohod metric, then the following holds:

For every $\epsilon > 0$ and $T \geq 0$, there exists a compact set $\Gamma \subset E$, such that

$$\mathbb{P}\Big[X_n(t) \in \Gamma \quad \text{for all } 0 \leq t \leq T\Big] \geq 1 - \epsilon.$$