**Institute for Applied Mathematics**
**WS 2020/21**
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# Sheet 9, "Introduction to Stochastic Analysis"

**Due on January 15, 2021**

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**Remark on informal stochastic calculus:**
To compute the cross-variation $[X,Y]$ of two processes $X$ and $Y$, one can use the following rule:

$$d[X,Y] = \begin{cases} 0 & \text{if either } X \text{ is of bounded variation or } Y, \\ dX \cdot dY & \text{else.} \end{cases}$$

Think about, why this works!

**Exercise 1** *[3 Pt]*

Show that there exists a local martingale, which is not a martingale.

**Exercise 2** *[6 Pt]*

Let $W$ be a standard Brownian motion and let the process $\Gamma$ be the solution of

$$\Gamma_0 = 1, \quad d\Gamma_t = \Gamma_t \left(\beta_t dt + \gamma_t dW_t\right).$$

Here $\beta$ and $\gamma$ are bounded, adapted processes. Assume that there is a $c > 0$ such that $\gamma_s > c$ for all $s$. Finally, let $T \in (0, \infty)$.

(a) Show that $\Gamma_t \exp\left(-\int_0^t \beta_s ds\right)$ is a local martingale.

(b) Find a probability measure $\mathbb{Q}_T$ s.t. $(\Gamma_t)_{t \leq T}$ is a local martingale under $\mathbb{Q}_T$.

(c) Compute $d\Gamma_t^{-1}$.

(d) Find a probability measure $\mathbb{R}_T$ s.t. $(\Gamma_t^{-1})_{t \leq T}$ is a local martingale under $\mathbb{R}_T$.

**Exercise 3** *[5 Pt]*

Let $B$ be standard Brownian motion on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and $\mathbb{P}^b$ a measure defined on $\mathcal{F}_T$ through

$$d\mathbb{P}^b = \exp\left(-b\int_0^T B_s dB_s - \frac{b^2}{2}\int_0^T B_s^2 ds\right) d\mathbb{P}.$$

You may assume that $\mathbb{P}^b$ is a probability measure (why is this true?).

(a) Show that the process

$$W_t = B_t + b\int_0^t B_s ds, \quad 0 \le t \le T$$

is a $\mathbb{P}^b$-Brownian motion.

(b) Show that

$$\int_0^t B_s dB_s = \frac{1}{2}(B_t^2 - t),$$

$\mathbb{P}^b$-almost surely.

(c) Show that for all $t \le T$:

$$\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\alpha B_t^2 - \frac{b^2}{2}\int_0^t B_s^2 ds\right)\right] = \mathbb{E}_{\mathbb{P}^b}\left[\exp\left(-\alpha B_t^2 + \frac{b}{2}(B_t^2 - t)\right)\right].$$

**Exercise 4** *[6 Pt]*

Let $B$ be a standard Brownian motion.

(a) Let $f$ be a continuous function on $[0,1]$ and consider

$$Z := \int_0^1 f(s) dB_s\,.$$

Show that $Z$ is a Gaussian random variable and compute the variance.

(b) Let $m \in N$ and define the following stochastic integrals:

$$A_m = \sqrt{2}\int_0^1 \cos(2\pi m t) dB_t, \quad B_m = \sqrt{2}\int_0^1 \sin(2\pi m t) dB_t.$$

Show that the following holds true for any $m \ge 1$:

(i) $A_m, B_m \sim \mathcal{N}(0,1)$.

(ii) $A_m$ and $B_m$ are uncorrelated.

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**Invitation from the working group on equality in Mathematics:** We invite all female, non-binary, intersexual and trans math students to join us in the online networking event **Tea Time with Women in Mathematics** on January 16th, 3-6 pm. You will have the opportunity to connect to other students as well as to ask questions to female Mathematicians at more advanced career stages over a cup of tea. Please register **here**. We are looking forward to meeting you!