Institute for Applied Mathematics
WS 2020/21
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Sheet 8, "Introduction to Stochastic Analysis"

Due on January 8, 2021

Exercise 1 *[6 Pt]*

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a standard filtered probability space, B a one-dimensional Brownian motion, and σ_t an adapted process such that $\mathbb{E}[\int_0^\infty \sigma_s^2 \mathrm{d}s] < \infty$. Define the process

$$Y_t = \exp\left(\int_0^t \sigma_s \mathrm{d}B_s - \frac{1}{2}\int_0^t \sigma_s^2 \mathrm{d}s\right). \tag{1}$$

(a) Use Ito's formula to show that

$$dY_t = \sigma_t Y_t dB_t$$

(b) Prove that $(Y_t)_{t\geq 0}$ is a supermartingale.

(c) If σ_t is constant, i.e. $\sigma_t = \sigma$, prove that Y_t is a martingale.

Exercise 2 *[7 Pt]*

Let B be a d-dimensional Brownian motion starting at $x \neq 0$. For $a > 0$ define the stopping time $T_a = \inf\{t \geq 0 : \ |B_t| = a\}$.

1. Let $d = 2$ and $0 < r < |x| < R$. Show that $\log(|B_{t\wedge T_r \wedge T_R}|)$ is a bounded martingale and prove that
$$\mathbb{P}[T_r < T_R] = \frac{\log R - \log|x|}{\log R - \log r}.$$
Deduce that B never hits the origin a.s.

2. Let $d = 3$. Show that $|B_{t\wedge T_r\wedge T_R}|^{-1}$ is a bounded martingale and that
$$\mathbb{P}[T_r < T_R] = \frac{R^{-1} - |x|^{-1}}{R^{-1} - r^{-1}}.$$
Deduce that $\mathbb{P}[T_r < \infty] = r|x|^{-1}$.

Exercise 3 *[7 Pt]*

1. Let M be a continuous local martingale. Show that for all $a < b$ and on a set of probability one,

$$[M]_a(\omega) = [M]_b(\omega) \Leftrightarrow \forall_{t\in[a,b]}: \; M_t(\omega) = M_a(\omega).$$

2. Consider two independent, continuous martingales M, N. Show that $[M, N] = 0$.
Hint: If $(\mathfrak{F}_t)_t$ and $(\mathfrak{G}_t)_t$ are the canonical filtrations of M and N respectively, then

$$\sigma(\mathfrak{F}_t \cup \mathfrak{G}_t) = \sigma\left(\left\{A_1 \cap A_2 \middle| A_1 \in \mathfrak{F}_t, A_2 \in \mathfrak{G}_t\right\}\right).$$

Remarks: (1) This implies that $[B^i, B^j] = \delta_{ij} t$, if $B = (B^1, \ldots, B^d)$ is a $d-$dimensional Brownian motion. (2) The statement holds also for local martingales.

The stochastic analysis team wishes you a merry Christmas and a good start into the new year!

The following exercises will neither be graded nor discussed in detail during the exercise classes. They should encourage you to not visit your whole family and instead think about the solutions during the holidays :)

Exercise 4 *[0 Pt]*

Let B be a one-dimensional Brownian motion and $h \in L^2([0,1], \lambda)$ be a deterministic function. Consider the Itō-integral

$$I_t := \int_0^t h(s) dB_s \qquad \text{for } 0 \le t \le 1.$$

1. Show that I_t is normally distributed with mean zero and variance

$$\tau(t) = \int_0^t h(r)^2 dr.$$

2. Prove that the increments of $(I_t)_{t\in[0,1]}$ are independent with law

$$I_t - I_s \sim N(0, \tau(t) - \tau(s)) \qquad \text{for } 0 \le s \le t.$$

3. Conclude that the process $(I_t)_{t\in[0,1]}$ has the same law as the **time-changed Brownian motion** $t \to B_{\tau(t)}$.

Exercise 5 *[0 Pt]*

Let $\mathbb{P}$ and $\mathbb{Q}$ be probability measures on $(\Omega, \mathcal{A})$ such that $\mathbb{Q}$ is absolutely continuous w.r.t. $\mathbb{P}$ (for all $A \in \mathcal{A}$: if $\mathbb{P}(A) = 0$ then also $\mathbb{Q}(A) = 0$). We will show a version of the famous **Radon-Nikodym theorem** using the martingale theory we have learned so far, proving the existence of a relative density under the assumption of absolute continuity. A relative density of $\mathbb{Q}$ w.r.t $\mathbb{P}$ is a measurable random variable $Z : \Omega \to [0, \infty)$, such that for all $A \in \mathcal{A}$:

$$\mathbb{Q}(A) = \int_A Z(\omega) d\mathbb{P}(\omega).$$

Let $\mathcal{A} = \sigma(\cup_n \mathcal{F}_n)$, where $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is a filtration of $\mathcal{A}$ consisting of finite σ-algebras $\mathcal{F}_n$, i.e. $\mathcal{F}_n = \sigma(B_{n,1}, \ldots, B_{n,k_n})$ such that $\cup_i B_{n,i} = \Omega$.

1. Write down a relative density Z_n of $\mathbb{Q}$ w.r.t $\mathbb{P}$ on $\mathcal{F}_n$ and show that $(Z_n)_{n\in\mathbb{N}}$ is a non-negative martingale under $\mathbb{P}$.

2. Show that the limit Z_∞ exists $\mathbb{P}$-almost surely and in $L^1(\Omega, \mathcal{A}, \mathbb{P})$.

3. Conclude that Z_∞ is a relative density of $\mathbb{Q}$ w.r.t $\mathbb{P}$ on $\mathcal{A}$.