

Sheet 3, “Introduction to Stochastic Analysis”

Due on November 20, 2020

Exercise 1

[5 Pt]

Let $(X_t)_{t \geq 0}$ be a non-negative, right-continuous submartingale w.r.t to the filtration $(\mathcal{F}_t)_{t \geq 0}$. Show that the following statements are equivalent:

1. The family $(X_t, t \in \mathbb{R}_+)$ is uniformly integrable.
2. The L_1 -limit $\lim_{t \rightarrow \infty} X_t$ exists.
3. There exists a uniformly integrable r.v. X_∞ such that
 - $\lim_{t \rightarrow \infty} X_t = X_\infty$ a.s.
 - $(X_t, \mathcal{F}_t)_{t \in \mathbb{R}_+ \cup \{+\infty\}}$ is a submartingale, where $\mathcal{F}_\infty := \sigma(\cup_{t \in \mathbb{R}_+} \mathcal{F}_t)$

Exercise 2

[4 Pt]

Let Z be a square integrable random variable on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, \mathbb{P})$. We define the stochastic process $(M_t)_{t \in \mathbb{R}_+}$ by

$$M_t = \mathbb{E}[Z | \mathcal{F}_t].$$

Prove that:

1. $(M_t)_{t \in \mathbb{R}_+}$ is a martingale.
2. $\mathbb{E}[\sup_{t \in \mathbb{R}_+} M_t^2] \leq 4\mathbb{E}[Z^2]$.

Exercise 3

[6 Pt]

Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, \mathbb{P})$ and $1 < p < \infty$. For a process $(M_t)_{t \in \mathbb{R}_+}$ we define $\|M\| = (\sup_{t \in \mathbb{R}_+} \mathbb{E}[|M_t|^p])^{1/p}$. Furthermore, consider the space

$$\mathcal{M}^p = \{(M_t)_{t \in \mathbb{R}_+} \text{ is a martingale, } M_0 = 0, \|M\| < \infty\}.$$

1. Show that $(\mathcal{M}^p, \|\cdot\|)$ is a Banach space.

2. Show that the map

$$\begin{aligned} \mathbf{j} : \mathcal{M}^p &\rightarrow L^p(\Omega, \mathcal{F}, \mathbb{P}) \\ (M_t)_{t \in \mathbb{R}_+} &\mapsto M_\infty \end{aligned}$$

is an isometry between Banach spaces.

Exercise 4

[5 Pt]

Fix $t > 0$. Let for all $n \in \mathbb{N}$, $\Pi^n := \{0 = t_0 < t_1 < \dots < t_n = t\}$ be a sequence of partitions of $[0, t]$ such that $\lim_{n \rightarrow \infty} \|\Pi^n\| = 0$, where $\|\Pi^n\| := \max_{1 \leq k \leq n} |t_k - t_{k-1}|$. For $q > 0$ consider the q -Variation of a process X w.r.t. Π^n defined as

$$V_t^{(q)}(\Pi^n) := \sum_{k=1}^n |X_{t_k} - X_{t_{k-1}}|^q.$$

Assume that X is a continuous, adapted process s.t. for some $p > 0$

$$\lim_{n \rightarrow \infty} V_t^{(p)}(\Pi^n) = L_t^p \text{ a.s.},$$

for a random variable L_t^p , which takes values on $[0, \infty]$ a.s.

1. Let $0 < q < p$ and assume that $L_t^p > 0$ a.s. Show that $\lim_{n \rightarrow \infty} V_t^{(q)}(\Pi^n) = \infty$ a.s.
2. Let $q > p$ and assume that $L_t^p < \infty$ a.s. Show that $\lim_{n \rightarrow \infty} V_t^{(q)}(\Pi^n) = 0$ a.s.