Institute for Applied Mathematics
WS 2020/21
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Sheet 11, “Introduction to Stochastic Analysis”

Due on January 29, 2021

Exercise 1 *[10 Pt]*

Let B_t be a three-dimensional Brownian motion, starting at $x \neq 0$. Prove the following statements:

i) $(X_t)_{t\geq 0}$ is a local martingale, where $X_t := |B_t|^{-1}$.

ii) $\mathbb{E}[\sup_{t\geq 0} X_t] = \infty$.
Hint: $\mathbb{E}[Z] = \int_0^\infty \mathbb{P}[Z \geq y]dy$ for $Z \geq 0$.

iii) The family $(X_t)_{t\geq 0}$ is uniformly integrable.
Hint: You may use the inequality

$$\sup_{t\geq 0} \mathbb{E}\left[X_t^p\right] \leq c \int_{\mathcal{B}} \frac{1}{|y|^p} dy + 1\,,$$

which holds for all $p > 0$ and the unit ball $\mathcal{B} = \{z \in \mathbb{R}^3 :\ |z| < 1\}$.

iv) $(X_t)_{t\geq 0}$ is not a martingale. Towards this goal, prove that

$$\mathbb{E}[X_t^p] \leq \frac{1}{2t^{p/2}} \int_0^\infty y^{-p/2} e^{-y/2} dy.$$

Hint: Chi-squared distribution!

v) Conclude that $\lim_{t\to\infty} |B_t| = \infty$ almost surely.

Exercise 2 *[3 Pt]*

Let B_t be a one-dimensional Brownian motion. Find the SDEs satisfied by the following processes:

1. $X_t = B_t/(1+t)$ for all $t \geq 0$,
2. $X_t = \sin(B_t)$ for all $t \geq 0$,

3. $(X_t, Y_t) = (a\cos(B_t), b\sin(B_t))$ for all $t \geq 0$, where $a, b \in \mathbb{R}$ with $ab \neq 0$.

Exercise 3 *[4 Pt]*

Let B_t be a one-dimensional Brownian motion and $\xi \in \mathbb{R}$. Furthermore, let $A(t), a(t)$ and $\sigma(t)$ be deterministic, measurable and locally bounded. Show that the SDE

$$dX_t = [A(t)X_t + a(t)]dt + \sigma(t)dB_t, \; X_0 = \xi,$$

admits the unique solution

$$X_t = \phi(t)\left(\xi + \int_0^t \phi^{-1}(s)a(s)ds + \int_0^t \phi^{-1}(s)\sigma(s)dB_s\right), \; 0 \leq t < \infty,$$

where ϕ is the unique solution of the ODE

$$\frac{d}{dt}\phi(t) = A(t)\phi(t), \; \phi(0) = 1.$$

Exercise 4 *[3 Pt]*

Suppose S_t is a solution of the SDE $dS_t = S_t(b_t dt + \sigma_t dB_t)$ (modeling a stock), where B_t is a one-dimensional Brownian motion. Let b_t, σ_t and r_t be deterministic, locally bounded functions. Further, assume that $\sigma > c$ for some $c > 0$.

1. Let $T > 0$ be finite. Show that there is a probability measure $\mathbb{Q}_{\mathbb{T}}$ such that for $t < T$ and under $\mathbb{Q}_T$:
$$dS_t = S_t(r_t dt + \sigma_t d\widetilde{B}_t),$$
where $\widetilde{B}$ is a $\mathbb{Q}_T$-Brownian motion (up to time T).

2. Set $R_t := \exp\left(-\int_0^t r(s)ds\right)$. Show that $R_t S_t$ is a martingale under $\mathbb{Q}_T$.