Institut für Angewandte Mathematik
SS 2022
Prof. Dr. Anton Bovier,
Florian Kreten

Stochastic processes

Sheet 13: repetition and mock exam

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a filtered probability space.

Exercise 1

1. Give the definition (or an equivalent characterization) of a familiy of uniformly integrable random variables.
2. Let $\mathcal{C}$ be a family of random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $g : [0, \infty) \to (0, \infty)$ be such that $g(r)/r \to \infty$ as $r \to \infty$. Let $\sup_{X \in \mathcal{C}} \mathbb{E}[g(|X|)] < \infty$. Show that $\mathcal{C}$ is uniformly integrable.
3. Let $(X_n)_{n\in\mathbb{N}}$ be a martingale adapted to the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$. Let T be a stopping time such that $\mathbb{E}[|X_T|] < \infty$ and $\lim_{n\to\infty} \mathbb{E}[|X_n|\mathbb{1}_{T>n}] = 0$. Show that $(X_{T\wedge n})_{n\in\mathbb{N}}$ is uniformly integrable.

Exercise 2

Let X be a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra.

1. Give the definition of the conditional expectation of X given $\mathcal{G}$.
2. Let T_1 and T_2 be independent exponential random variables with parameter $\alpha > 0$ and let $X = \min(T_1, T_2)$. Compute $\mathbb{E}[X|T_1]$.

   Remark. Recall that the probability density function of an exponentially distributed random variable with parameter $\alpha > 0$ is given by $r \mapsto \alpha e^{-\alpha r}\mathbb{1}_{[0,\infty)}(r)$.

Exercise 3

1. Let T be a stopping time. Define the pre-T-σ-algebra $\mathcal{F}_T$.
2. Let $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ be martingales adapted to the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$. Let T be a stopping time such that $X_T = Y_T$. Let for all $n \in \mathbb{N}$, $Z_n = X_n\mathbb{1}_{\{T\leq n-1\}} + Y_n\mathbb{1}_{\{T\geq n\}}$. Show that $(Z_n)_{n\in\mathbb{N}}$ is a martingale adapted to the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$.

3. Let $(B_t)_{t\in\mathbb{R}_+}$ be the one-dimensional Brownian motion starting in 0. Let $\sigma > 0$. Show that $(\mathrm{e}^{\sigma B_t - \frac{1}{2}\sigma^2 t})_{t\in\mathbb{R}_+}$ is a martingale.

Exercise 4

1. State Doob's convergence theorem for super-martingales.

2. Let $p > 1$ and let $(X_n)_{n\in\mathbb{N}}$ be a martingale such that $\sup_{n\in\mathbb{N}} \mathbb{E}[|X_n|^p] < \infty$. Show that there exists a random variable X_∞ such that $\mathbb{E}(|X_\infty|^p) < \infty$ and $\lim_{n\to\infty} X_n = X_\infty$ a.s. and in L^p.

3. Let $(Y_n)_{n\in\mathbb{N}_0}$ be independent standard normal random variables. Let $S_n = \sum_{i=1}^n Y_i$ and $X_n = \mathrm{e}^{S_n - \frac{n}{2}}$ for $n \geq 1$. Prove that $\lim_{n\to\infty} X_n = 0$ a.s.

Exercise 5

Let X_n, $n \in \mathbb{N}$ be i.i.d random variables, with $\mathbb{P}(X_1 = 1) = 1/2 = \mathbb{P}(X_1 = -1)$. Let $\alpha \in \mathbb{N}$ (which implies that $\alpha > 0$). Let $S_0 = \alpha$ and $S_n = \alpha + \sum_{k=1}^n X_k$ for $n \in \mathbb{N}$.

1. Show that $\{S_n\}_{n\in\mathbb{N}\cup\{0\}}$ is a Markov process.

2. Compute the generator of S_n.

3. Use the martingale problem to show that $\{\frac{1}{3}S_n^3 - \sum_{l=0}^n S_l\}_{n\in\mathbb{N}\cup\{0\}}$ is a martingale.

4. Let $\alpha < K \in \mathbb{N}$. Define the hitting times

$$\tau_0 := \inf\{n > 0 \mid S_n = 0\} \qquad \text{and} \qquad \tau_K := \inf\{n > 0 \mid S_n = K\}.$$

   Set $\tau = \tau_0 \wedge \tau_K$. Show that $\mathbb{E}[\sum_{l=0}^{\tau} S_l] = \frac{1}{3}(K^2 - \alpha^2)\alpha + \alpha$.