Institute for Applied Mathematics
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Stochastic Processes
Sheet 12

To hand in via ecampus before Friday, July 01

Exercise 1

The Wright-Fisher model describes the evolution of a population of individuals with phenotype A or B, where the total population size is assumed to be constant over time and equal to $N \in \mathbb{N}$. We let $(X_n)_{n\in\mathbb{N}_0}$ with state space $S = \{0, \ldots, N\}$ be a stochastic process in discrete time, such that

$$X_n := \text{number of individuals of type } A \text{ in the } n\text{th generation.}$$

Thus, the number of individuals of type B is equal to $Y_n = N - X_n$. In this model, the evolution is given as follows: Given the generation at time n, each of the N individuals of the generation at time $n+1$ takes the type from a randomly (with a uniform distribution) chosen parent of the generation at time n. Show that

1. $(X_n)_n$ is a Markov Chain and compute the transition probability,
2. $(X_n)_n$ is a martingale,
3. the states 0 and N are absorbing,
4. $\mathbb{E}_x\left[\tau_{\{0,N\}}\right] < \infty$ for all $x \in \{0, \ldots, N\}$,
5. Compute (using e.g. Doob's Optional Stopping Theorem) the probability that the process started at $x \in \{0, \ldots, N\}$ is absorbed in N (resp. in 0).

Exercise 2

Consider the simple random walk on $\{-N, -N+1, \ldots, N\}$. Let for $x \in \{-N, -N+1, \ldots, N\}$

$$h(x) = \mathbb{P}_x[\tau_N = \tau_{\{N\}\cup\{-N\}}] = \mathbb{P}_x[\tau_N < \tau_{-N}].$$

Assume we want to condition this process on hitting $+N$ before $-N$. Compute $h(x)$ and use this to compute the transition rates of the h-transformed walk

Exercise 3

Let $(X_n)_n$ be a Markov Chain taking values in $\mathbb{N}_0$ and with transition matrix P given by

$$\begin{aligned} &P(0,0) = 1, \\ &P(k,m) = e^{-k}\frac{k^m}{m!} \qquad \text{for } k \in \mathbb{N} \text{ and } m \in \mathbb{N}_0. \end{aligned}$$

1. Which states are recurrent?
2. Show that the identity function $f : \mathbb{N}_0 \to \mathbb{N}_0$, $k \mapsto k$, is harmonic.

Exercise 4

Let $(B_t)_{t\in\mathbb{R}_+}$ be the Brownian motion. Define the processes $(B_t^{(1)})_{t\in\mathbb{R}_+}, (B_t^{(2)})_{t\in\mathbb{R}_+}, (B_t^{(3)})_{t\in\mathbb{R}_+}$ by

1. $B_t^{(1)} = -B_t$,
2. $B_t^{(2)} = B_{t+r} - B_r$ for some $r > 0$,
3. $B_t^{(3)} = \frac{1}{c}B_{c^2t}$ for some $c > 0$.

Show that $(B_t^{(1)})_{t\in\mathbb{R}_+}, (B_t^{(2)})_{t\in\mathbb{R}_+}, (B_t^{(3)})_{t\in\mathbb{R}_+}$ are Brownian motions as well.