Institute for Applied Mathematics
SS 2022
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Stochastic Processes
Sheet 11

To hand in via ecampus before Friday, June 24

For your admission to the exams (but not regarding their content), this is the last relevant sheet. More information in the next tutorials.

Exercise 1 [*7 Pts.*]

Let $(X_n)_{n\in\mathbb{N}}$ be i.i.d. random variables with $\mathbb{P}(X_1 = -1) = \mathbb{P}(X_1 = +1) = \frac{1}{2}$. Let $S_0 = 0$ and let $S_n = \sum_{i=1}^n X_i$ for all $n \geq 1$. Further, define for $a, b \in \mathbb{N}$ the following hitting times

$$\tau_{-a} = \inf\{n > 0 \mid S_n = -a\} \qquad \text{and} \qquad \tau_b = \inf\{n > 0 \mid S_n = b\}.$$

Set $\tau = \tau_{-a} \wedge \tau_b$. Prove that

1. $\mathbb{E}(\tau) < \infty$,
2. $(S_n^2 - n)_n$ is a martingale,
3. $\mathbb{E}(S_\tau) = 0$,
4. $\mathbb{P}(\tau_{-a} < \tau_b) = \frac{b}{a+b}$, (*Hint*: Use the Optional Stopping Theorem!)
5. $\mathbb{E}(\tau) = \mathbb{E}(S_\tau^2)$.

Finally, compute $\mathbb{E}(\tau)$.

Exercise 2 [*5 Pts.*]

Suppose that $P(x, dy)$ is a transition kernel on a state space $(S, \mathcal{B})$. We say that a probability measure μ on $(S, \mathcal{B})$ satisfies the *detailed balance condition w.r.t. P* if and only if for all measurable $f : S \times S \to \mathbb{R}_+$,

$$\int\int \mu(dx)\, P(x, dy)\, f(x, y) = \int\int \mu(dy)\, P(y, dx)\, f(x, y).$$

a) Show that a measure that satisfies the detailed balance condition is invariant.

b) Suppose that $(X_n)_{n\in\mathbb{N}}$ is a stationary Markov chain with one step transition kernel P and with initial distribution μ. Show that for all $n \geq 0$, the distribution of X_n is equal to μ.

c) Now let $p \in (0,1)$, and consider a Markov chain with state space $\mathbb{Z}_+$ and transition probabilities $P(x, x+1) = p$ for $x \geq 0$, $P(x, x-1) = q := 1-p$ for $x \geq 1$, and $P(0,0) = q$.

   (i) Find a nontrivial invariant measure.

   (ii) Show that if $p < q$ then there is a unique invariant probability measure.

   (iii) Show that if $p \geq q$ then an invariant probability measure does not exist.

Exercise 3 *[3 Pts.]*

Show that a Markov chain with stationary transition kernel P and initial distribution $\mathbb{P}_0 = \pi$ is a stationary stochastic process if and only if π is an invariant probability distribution under P.

Exercise 4 *[5 Pts.]*

Let $(X_n)_{n\in\mathbb{N}}$ be a Markov chain taking values in $E = \{0, 1, \ldots, N\}$ and with transition matrix P given by

$$P(x,y) = \begin{cases} p & \text{if } y = x+1 \\ 1-p & \text{if } y = 0 \\ 0 & \text{otherwise} \end{cases}$$

for $0 \leq x \leq N-1$ and for $0 < p < 1$. Moreover, let the state N be *absorbing*: $P(N,N) = 1$. Define $\tau = \inf\{n \geq 0 : X_n = N\}$, i.e. the first time that X reaches N.

1. Use the Markov property to prove that $u(x) = \mathbb{E}_x[\tau], x \in \{0, 1, \ldots, N\}$ satisfies the following equation

$$u(x) = \begin{cases} 0 & \text{if } x = N \\ 1 + \sum_{y\in E} P(x,y)u(y) & \text{otherwise} \end{cases}$$

2. Compute $\mathbb{E}_x[\tau]$.

3. How many tosses of a fair coin are necessary on average to get six heads in a row?