Institute for Applied Mathematics
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Stochastic Processes
Sheet 9

To hand in via ecampus before Friday, June 03

Exercise 1 *[5 Pts.]*

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let ν be a finite measure on $(\Omega, \mathcal{F})$ such that $\nu \ll \mathbb{P}$. Let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be a filtration and for all $n \in \mathbb{N}$, let X_n be the Radon-Nikodým derivative of ν with respect to $\mathbb{P}$ on $(\Omega, \mathcal{F}_n)$. Show that $(X_n)_{n\in\mathbb{N}}$ is a martingale.

Exercise 2 *[5 Pts.]*

Let $(Y_n)_{n\in\mathbb{N}_0}$ be independent standard normal random variables. Let $S_n = \sum_{i=1}^n Y_i$ and $X_n = e^{S_n - \frac{n}{2}}$ for $n \geq 1$. Prove that

1. $(X_n)_{n\in\mathbb{N}_0}$ is a martingale,
2. $\lim_{n\to\infty} X_n = 0$ a.s,
3. $\lim_{n\to\infty} \mathbb{E}[X_n^p] = 0$, if and only if $p < 1$. (Hence, although the a.s. limit of $(X_n)_n$ is in L^1, $(X_n)_n$ does not converge to zero in L^1.)

Exercise 3 *[5 Pts.]*

Let X_n, $n \in \mathbb{N}$ be a sequence of random variables. Define $\mathcal{T}_n = \sigma(X_k, k \geq n)$ and $\mathcal{T} = \bigcap_{n\in\mathbb{N}} \mathcal{T}_n$. The σ-algebra $\mathcal{T}$ is called the *tail σ-algebra*, and elements from $\mathcal{T}$ are called *tail events*. Furthermore, call a $\mathbb{R} \cup \{\pm\infty\}$-valued random variable η *degenerate*, if there exists a $c \in \mathbb{R} \cup \{\pm\infty\}$, such that $\eta = c$ a.s.

1. Which of the following events are $\mathcal{T}$ measurable?
$$\Big\{\lim_{n\to\infty} X_n \text{ exists}\Big\}, \qquad \Big\{\sup_n X_n < c\Big\}, \qquad \Big\{\limsup_{n\to\infty} X_n < c\Big\}$$
$$\Big\{\sum_{n=1}^{\infty} X_n \text{ converges}\Big\}, \qquad \Big\{\sum_{n=1}^{\infty} |X_n| < c\Big\}.$$
2. Now suppose that X_n, $n \in \mathbb{N}$ is a sequence of independent random variables. Let $S_n = \sum_{i=1}^{n} X_i$. Show that $\limsup_{n\to\infty} X_n$ and $\limsup_{n\to\infty} \frac{S_n}{n}$ are degenerate random variables.

Exercise 4 *[5 Pts.]*

Let $(Y_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. random variables on the probability space $(\Omega, \mathcal{F}, P)$, where Y_1 is not degenerate. Let $X_n := \sum_{k=1}^n Y_k$ and

$$\begin{aligned}\phi &: \mathbb{R} \to \mathbb{R} \cup \{\infty\}, \quad \phi(u) = \log E[\exp(uY_1)],\\ \mathcal{U} &:= \{u \in \mathbb{R} \mid \phi(u) \in \mathbb{R}\}.\end{aligned}$$

1. Find a function $g : \mathbb{N}\times\mathcal{U} \to \mathbb{R}$, such that $M_n(u) := \exp(uX_n - g(n, u))$ is a martingale for all $u \in \mathcal{U}$.

2. Explain why $(M_n(u))_n$ with $u \in \mathcal{U}$ converges almost surely and verify that $0 \in \mathcal{U}$. Show that for $u \neq 0$, $\phi(tu) < t\phi(u)$ for all $t \in (0, 1)$ and that the martingale $(M_n(u))_n$ converges almost surely to zero.

   Hint: To show $\phi(tu) < t\phi(u)$, ask yourself which are the only cases where equality holds in Jensen's inequality!