Institute for Applied Mathematics
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Stochastic Processes
Sheet 8

To hand in via ecampus before Friday, May 27

Exercise 1 *[6 Pts.]*

Consider the "Brownian motion" process X defined in Section 3.3.2 from the Lecture Notes. Show that this process is Markov in the sense that all finite dimensional distributions satisfy the Markov property. That is, show that for all $N \in \mathbb{N}$, $0 \leq t_1 < \ldots < t_N < \infty$ and all measurable and bounded functions $f : \mathbb{R} \to \mathbb{R}$:

$$\mathbb{E}[f(X_{t_N})|\sigma(X_{t_{N-1}}, \ldots, X_{t_1})] \;=\; \mathbb{E}[f(X_{t_N})|X_{t_{N-1}}] \quad \text{a.s.}$$

Hint: Let $Y_N = X_{t_N} - X_{t_{N-1}}$. Show first that the random vector $Y_N, X_{t_{N-1}}, \ldots X_{t_1}$ is Gaussian and that Y_N is independent of $\sigma(X_{t_{N-1}}, \ldots X_{t_1})$.

Exercise 2 *[3 Pts.]*

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $X = (X_n)_{n\in\mathbb{N}_0}$ be a martingale with respect to the filtration $\{\mathcal{G}_n : n \in \mathbb{N}_0\}$. Let $\mathcal{F}_n = \sigma(\{X_k : 0 \leq k \leq n\})$. Show that $\mathcal{F}_n \subset \mathcal{G}_n$ for all $n \in \mathbb{N}_0$ and that X is a martingale with respect to the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$.

Exercise 3 *[7 Pts.]*

Let X, Y be martingales with respect to a filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$. Show that

(a) For $a, b \in \mathbb{R}$, $aX + bY$ is a martingale,

(b) $\max\{X, Y\}$ is a submartingale,

(c) $\min\{X, Y\}$ is a supermartingale.

(d) Let ϕ a convex function such that $\mathbb{E}|\phi(X_n)| < \infty$ for all n. Then $\big(\phi(X_n)\big)_{n\in\mathbb{N}_0}$ is a submartingale.

(e) Let Z be a submartingale and ϕ a convex non-decreasing function with $\mathbb{E}|\phi(Z_n)| < \infty$ for all n. Then $\big(\phi(Z_n)\big)_{n\in\mathbb{N}_0}$ is a submartingale.

Exercise 4 *[4 Pts.]*

Let $X_k, k \in \mathbb{N}$ be independent and identically distributed random variables with $\mathbb{E}[X_1] = 0$ and $\text{Var}[X_1] = 1$. For $n \in \mathbb{N}$ and $t \in [0,1]$ let

$$Z_n(t) := \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} X_k,$$

such that for all $n \in \mathbb{N}$, $\big(Z_n(t)\big)_{t \in [0,1]}$ is a stochastic process with state space $\mathbb{R}$.

Show that the finite dimensional distributions of Z_n converge in law to the "Brownian motion" as $n \to \infty$ (see Section 3.3.2 from the Lecture Notes): for every partition $0 \leq t_1 < \ldots < t_N \leq 1$, $N \in \mathbb{N}$, the vectors $\big(Z_n(t_1), \ldots Z_n(t_N)\big)$ converge in law to an N-dimensional Gaussian random vector with mean zero and covariance matrix C, such that $C_{ij} = t_i \wedge t_j$.

Hint: Compare to exercise 4 on sheet 7. The following identity might be useful:

$$\begin{pmatrix} Z_n(t_1) \\ Z_n(t_2) \\ \vdots \\ Z_n(t_N) \end{pmatrix} = \begin{pmatrix} Z_n(t_1) \\ Z_n(t_1) + (Z_n(t_2) - Z_n(t_1)) \\ \vdots \\ Z_n(t_1) + \sum_{i=1}^{N-1} (Z_n(t_{i+1}) - Z_n(t_i)) \end{pmatrix}.$$