Institute for Applied Mathematics
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Stochastic Processes
Sheet 7

To hand in via ecampus before Friday, May 20

Remark. In what follows, you may use the following facts:

1. The law of an $\mathbb{R}^n$-valued random vector X is uniquely determined by its multi-dimensional characteristic function given by $\mathbb{R}^n \ni u \mapsto \mathbb{E}[e^{i\langle u,X\rangle}]$.

2. Let $(X_k)_{k\in\mathbb{N}}$ be a sequence of $\mathbb{R}^n$-valued random vectors with multi-dimensional characteristic functions given by ϕ_k. Let X be an $\mathbb{R}^n$-valued random vector with multi-dimensional characteristic function given by ϕ. If $\lim_{k\to\infty} \phi_k(u) = \phi(u)$ for all $u \in \mathbb{R}^n$, then the sequence $(X_k)_{k\in\mathbb{N}}$ converges in law to X.

Exercise 1 *[6 Pts.]*

Let X_1, X_2 be two jointly Gaussian random variables, i.e. two real valued random variables whose joint distribution is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^2$ with density

$$p(x_1,x_2) = \frac{\exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)\right]}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}.$$

1. Show that $\mathbb{E}[X_i] = \mu_i$ and $\mathrm{Var}[X_i] = \sigma_i^2$ for $i = 1, 2$ and $\mathrm{Cov}(X_1, X_2)/(\sigma_1\sigma_2) = \rho$[1]. For the computations, you can use all known facts about one-dimensional Gaussian random variables.

2. Find the conditional expectation $\mathbb{E}[X_1|X_2]$ and the conditional density $f_{X_1|X_2}$ for X_1 given X_2.

Exercise 2 *[4 Pts.]*

1. Let X be an $\mathbb{R}^n$-valued Gaussian vector with mean zero and covariance matrix C. Show that the components $X_1, \ldots, X_n$ of X are independent if and only if the covariance matrix C of X is diagonal.

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[1]For random variables X and Y, $\rho(X,Y) \equiv \mathrm{Cov}(X,Y)/\sqrt{\mathrm{Var}(X)\mathrm{Var(Y)}}$ is called the *correlation between X and Y*.

2. Let N be a real valued Gaussian random variable with mean zero and variance 1 and let Z be another random variable, independent of N, with $\mathbb{P}(Z = 1) = \mathbb{P}(Z = -1) = 1/2$. Let $Y = NZ$. Show that

   (a) Y is also a Gaussian random variable;

   (b) $\mathrm{Cov}(N, Y) = 0$, but N and Y are not independent;

   (c) (N, Y) is not jointly Gaussian.

Exercise 3 *[2 Pts.]*

Let X be an $\mathbb{R}^n$-valued Gaussian vector with mean zero and covariance matrix C. Let A be an invertible $n \times n$ matrix. Show that AX is a Gaussian vector as well, and compute its covariance matrix.

Hint: Compute $\mathbb{E}[F(AX)]$ for an arbitrary bounded and measurable function $\mathbb{R}^n \to \mathbb{R}$.

Exercise 4 *[6 Pts.]*

Let $X_k, k \in \mathbb{N}$ be independent Gaussian random variables with mean 0 and variance 1. For $n \in \mathbb{N}$ and $t \in [0, 1]$ let

$$Z_n(t) := \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} X_k,$$

where $[x]$ represents the largest integer smaller than x (i.e. $[\pi] = 3$). Hence, for all n, $(Z_n(t))_{t \in [0,1]}$ is a stochastic process with state space $\mathbb{R}$.

1. Compute for every partition $0 \leq t_1 < \ldots < t_N \leq 1$, $N \in \mathbb{N}$, the covariance matrix $(\mathrm{Cov}(Z_n(t_i), Z_n(t_j)))_{ij}$ and show that for $n \to \infty$ this matrix converges to a matrix C with $C_{ij} = t_i \wedge t_j$.

2. Show that the finite dimensional distributions of Z_n converge in law to the "Brownian motion" as $n \to \infty$ (see Section 3.3.2 from the Lecture Notes). This means that for every partition $0 \leq t_1 < \ldots < t_N \leq 1$, $N \in \mathbb{N}$, the vectors $(Z_n(t_1), \ldots Z_n(t_N))$ converge in law to an N-dimensional Gaussian random vector with mean zero and covariance matrix C where $C_{ij} = t_i \wedge t_j$.

Exercise 5 *[2 Pts.]*

Prove the Chapman-Kolmogorov equations (Lemma 3.18 in the lecture notes).