Institute for Applied Mathematics
SS 2022
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Stochastic Processes
Sheet 6

To hand in via ecampus before Friday, May 13

Exercise 1 *[5 Pts.]*

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra.

1. Prove the conditional Markov inequality, i.e. show that, if $f : \mathbb{R}_+ \to \mathbb{R}_+$ is non-decreasing and such that $f(|X|)$ is integrable, then

$$\mathbb{P}[|X| \geq \alpha|\mathcal{G}] \leq \frac{1}{f(\alpha)}\mathbb{E}[f(|X|)|\mathcal{G}] \quad \mathbb{P}\text{-a.s.}$$

2. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a convex function, X and $\phi(X)$ be integrable random variables. Prove the conditional Jensen inequality

$$\phi(\mathbb{E}[X|\mathcal{G}]) \leq \mathbb{E}[\phi(X)|\mathcal{G}].$$

Hint: Convexity of ϕ implies that for $x, y \in \mathbb{R}$, there exists a measurable function $c : \mathbb{R} \to \mathbb{R}$ such that

$$\phi(x) \geq \phi(y) + c(y)(x - y).$$

Exercise 2 *[4 Pts.]*

Let X be integrable and let Y be bounded and $\mathcal{G}$-measurable. By using the monotone class theorem, show that

$$\mathbb{E}(XY|\mathcal{G}) = Y\mathbb{E}(X|\mathcal{G}) \quad \text{a.s.}$$

Exercise 3 *[8 Pts.]*

1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{F}_0 \subset \mathcal{F}$ be a σ-Algebra. Let $F : \mathbb{R}^2 \to [0, \infty)$ be measurable, X be a random variable independent of $\mathcal{F}_0$, and Y_0 be an $\mathcal{F}_0$-measurable random variable. Show that

$$\mathbb{E}\big[F(X, Y_0)|\mathcal{F}_0\big](\omega) = \int_\Omega F(X(\omega'), Y_0(\omega))\,\mathbb{P}(d\omega') \qquad \text{a.s.}$$

2. Let T_1 and T_2 be independent exponential random variables with parameter $\alpha > 0$ and let $X = \min(T_1, T_2)$. Compute $\mathbb{E}[X|T_1]$.

Exercise 4 *[3 Pts.]*

Let $Y_1, Y_2, \dots$ be independent and identically distributed random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $Y_1 \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Further, let N be a non-negative integer valued random variable, independent of the Y_n's with $\mathbb{E}[N] < \infty$. Define the random variable $X = \sum_{k=1}^{N} Y_k$ and compute $\mathbb{E}[X]$.