Institute for Applied Mathematics
SS 2022
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Stochastic Processes
Sheet 5

To hand in via ecampus before Friday, May 06

Exercise 1 *[6 Pts.]*

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra. Let X and Y be absolutely integrable random variables.

1. Show that the map $X \to \mathbb{E}(X|\mathcal{G})$ is linear.
2. Show that if $\mathcal{B} \subset \mathcal{G}$ is a σ-algebra, then $\mathbb{E}[\mathbb{E}(X|\mathcal{G})|\mathcal{B}] = \mathbb{E}(X|\mathcal{B})$ a.s. (this is known as *tower-property*).
3. Show that if $X \leq Y$ a.s., then $\mathbb{E}(X|\mathcal{G}) \leq \mathbb{E}(Y|\mathcal{G})$ a.s.
4. Show that $|\mathbb{E}(X|\mathcal{G})| \leq \mathbb{E}(|X||\mathcal{G})$ a.s.;
5. Assume that there exists $n \in \mathbb{N}$ and $z_1, \ldots, z_n \in \mathbb{R}$ such that $\mathbb{P}(Y \in \{z_1, \ldots, z_n\}) = 1$ and $\mathbb{P}(Y = z_i) > 0$ for $i = 1, \ldots, n$. Compute $\mathbb{E}\left(X|\sigma(Y)\right)$.

Exercise 2 *[4 Pts.]*

Let $\Omega = \{\omega_1, \ldots, \omega_5\}$ and let $\mathcal{F} = 2^\Omega$ be the power set Ω. Let $\mathbb{P}$ be the unique probability measure such that

$$\mathbb{P}\big[\{\omega_1\}\big] = \tfrac{1}{10}, \quad \mathbb{P}\big[\{\omega_2\}\big] = \mathbb{P}\big[\{\omega_3\}\big] = \mathbb{P}\big[\{\omega_4\}\big] = \tfrac{1}{5}, \quad \mathbb{P}\big[\{\omega_5\}\big] = \tfrac{3}{10}.$$

Consider the σ-algebra $\mathcal{F}_1 = \sigma\big(\{\omega_1, \omega_4\}, \{\omega_5\}, \{\omega_2, \omega_3\}\big)$ and the random variable X defined by the following: $X(\omega_1) = 1$, $X(\omega_2) = 2$, $X(\omega_3) = 4$, $X(\omega_4) = 7$ and $X(\omega_5) = 12$. Compute $\mathbb{E}[X|\mathcal{F}_1]$.

Exercise 3 *[5 Pts.]*

Let $\mathcal{M}$ be the set of all measures on the measurable space $(\Omega, \mathcal{A})$.

1. Let $\mu \sim \nu$, if and only if $\mu \ll \nu$ and $\nu \ll \mu$. Show that $\mu \sim \nu$ is an equivalence relation.

2. Show that for finite measures μ and ν, $\mu \sim \nu$ is equivalent to $0 < \frac{\mathrm{d}\nu}{\mathrm{d}\mu} < \infty$ μ-a.e.

Exercise 4 *[5 Pts.]*

Let $(\Omega, \mathcal{A})$ be a measurable space, where the σ-algebra $\mathcal{A}$ contains all points, i.e. $\{\omega\} \in \mathcal{A}$ for all $\omega \in \Omega$. Let μ and ν be finite and discrete measures on $\mathcal{A}$.

1. Give a necessary and a sufficient condition for $\nu \ll \mu$.

2. Assume $\nu \ll \mu$. Calculate all densities of ν with respect to μ.

Definition: A measure μ is called *finite and discrete* if there are at most countably many $\omega_i \in \Omega$ and $p_i > 0$ with $\sum_i p_i < \infty$, such that

$$\mu = \sum_i p_i \delta_{\omega_i}.$$