Institute for Applied Mathematics
SS 2022
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Stochastic Processes
Sheet 4

To hand in via ecampus before Friday, April 29

Exercise 1 *[1+1+2+2+2 Pts.]*

Let $\mathcal{C}$ and $\mathcal{D}$ be classes of random variables.

1. Show that $\mathcal{C}$ is uniformly integrable, if and only if
$$\lim_{K\to\infty} \sup_{X\in\mathcal{C}} \mathbb{E}[|X| \cdot \mathbb{1}_{\{|X|>K\}}] = 0.$$
2. Show that $\mathcal{C} + \mathcal{D} := \{X + Y, X \in \mathcal{C}, Y \in \mathcal{D}\}$ is uniformly integrable, if $\mathcal{C}$ and $\mathcal{D}$ are uniformly integrable.
3. Let $g : [0, \infty) \to [0, \infty)$ be such that $g(x)/x \to \infty$ as $x \to \infty$. If $\sup_{X\in\mathcal{C}} \mathbb{E}[g(|X|)] < \infty$, show that $\mathcal{C}$ is uniformly integrable.
4. If there exists an integer $p > 1$ such that $\sup_{X\in\mathcal{C}} \mathbb{E}[|X|^p] < \infty$, show that $\mathcal{C}$ is uniformly integrable.
5. If $\mathbb{E}\big[\sup_{X\in\mathcal{C}} |X|\big] < \infty$, show that $\mathcal{C}$ is uniformly integrable.

Exercise 2 *[1+1+3 Pts.]*

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $Y, X, X_1, X_2, \ldots$ be random variables in $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$, such that $X_n \to X$ in $\mathcal{L}^2$. Show that

1. $\lim_{n\to\infty} \mathbb{E}[X_n Y] = \mathbb{E}[XY]$.
2. $\lim_{n\to\infty} \mathbb{E}[X_n^2] = \mathbb{E}[X^2]$.
3. Let $1 \leq p \leq q \leq \infty$. Show that $\mathcal{L}^p(\Omega, \mathcal{F}, \mathbb{P}) \supset \mathcal{L}^q(\Omega, \mathcal{F}, \mathbb{P})$. Why does this not hold on arbitrary measure spaces?

Exercise 3 [*4 Pts.*]

Use Jensen's inequality to prove the Hölder inequalties for $p, q \in [1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

Exercise 4 [*3 Pts.*]

Let μ be a finite measure on a measurable space $(\Omega, \mathcal{A})$. Let $f : \Omega \to \mathbb{R}$ be measurable, nonnegative and integrable. Define

$$\mu_f(A) = \int_A f \, d\mu \qquad \forall A \in \mathcal{A}.$$

Show that μ_f is a measure on $(\Omega, \mathcal{A})$.