Institute for Applied Mathematics
SS 2022
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Stochastic Processes
Sheet 3

To hand in via ecampus before Friday, April 22

Exercise 1 *[3 Pkt]*

Let $V = \{v : \mathbb{N} \to \mathbb{R} \mid \|v\| < \infty\}$ be the metric space of uniformly bounded sequences, where $\|v\| = \sup_{n\in\mathbb{N}} |v_n|$. Show that the closed unit sphere $B = \{v \in V \mid \|v\| = 1\}$ is not compact.

Exercise 2 *[3 Pkt]*

Show that each σ-finite measure μ on some measurable space $(\Omega, \mathcal{F})$ has a representation of the form $\mu = \sum_{n=0}^{\infty} a_n \mu_n$, where for all n: $a_n \geq 0$ is finite and μ_n is a probability measure on $(\Omega, \mathcal{F})$.

Exercise 3 *[3+1 Pkt]*

Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space and consider the associated outer measure

$$\mu^*(D) := \inf\{\mu(F) : F \in \mathfrak{F}, F \supseteq D\},$$

which is defined for all subsets of Ω. Assume that there exists a set $G \subset \Omega$ such that $\mu^*(G) = 1$, and let $\mathfrak{G} := \mathfrak{F} \cap G$ (all subsets of G of the form $G \cap A, A \in \mathfrak{F}$).

- Prove that for any $A \in \mathfrak{F} : \mu^*(G \cap A) = \mu(A)$.
- Conclude that $(G, \mathfrak{G}, \mu^*)$ is a probability space.

Exercise 4 *[3 Pkt]*

Let $f_n, n \in \mathbb{N}$ be real valued random variables. Prove that both

$$f^+ := \limsup_{n\to\infty} f_n \quad \text{and} \quad f^- := \liminf_{n\to\infty} f_n$$

are measurable.