Institute for Applied Mathematics
SS 2022
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Stochastic Processes
Sheet 10

To hand in via ecampus before Friday, June 17

Exercise 1 [*5 Pts.*]

1. Let $(X_n)_{n\in\mathbb{N}}$ be a submartingale such that $\mathbb{E}[e^{tX_n}] < \infty$ for all $n \in \mathbb{N}$ and for all $t \in \mathbb{R}$. Prove that
$$\mathbb{P}\left(\max_{1\le k\le n} X_k \ge x\right) \le e^{-tx}\mathbb{E}[e^{tX_n}],$$
for all $t > 0$ and $x \in \mathbb{R}$.

2. Let $(\xi_n)_{n\in\mathbb{N}}$ be i.i.d random variables with $\mathbb{P}(\xi_1 = 1) = 1/2 = \mathbb{P}(\xi_1 = -1)$. Let $X_n = \sum_{k=1}^n \xi_k$. Prove that
$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{X_n}{\sqrt{2n\log n}} \ge 1+\epsilon\right) = 0$$
for all $\epsilon > 0$.
Hint: Use $X_n \le \max_{1\le k\le n} X_k$, part 1., the fact that $\log\cosh x \le x^2/2$ and the Borel Cantelli lemmas.

Exercise 2 [*5 Pts.*]

Let $(X_n)_{n\in\mathbb{N}_0}$ be integrable and i.i.d. random variables with $\mathbb{E}(X_1) = 0$. For $\alpha \in (0,1)$ and $n \in \mathbb{N}_0$, define
$$M_n := \sum_{i=0}^n \alpha^i X_i.$$

- Show that $(M_n)_{n\in\mathbb{N}_0}$ is a martingale.
- Show that the sequence $(M_n)_{n\in\mathbb{N}_0}$ is bounded in L^1. Deduce that $(M_n)_{n\in\mathbb{N}_0}$ converges almost surely.
- Show that $(M_n)_{n\in\mathbb{N}_0}$ converges in L^1 and determine the limit M_∞. Further, show that $M_n = \mathbb{E}[M_\infty|\mathcal{F}_n]$.

Exercise 3 *[10 Pts.]*

Suppose that $(\Omega, \mathcal{F}, P)$ is a probability space and $(\mathcal{F}_n)_n$ is a filtration. Let τ, τ_1 and τ_2 be stopping times with respect to $(\mathcal{F}_n)_n$. Show that

1. In the definition of stopping times, one could require that $\{\tau_i \leq n\} \in \mathcal{F}_n$ for all n instead of $\{\tau_i = n\} \in \mathcal{F}_n$ for all n,
2. $\tau_1 + \tau_2$, $\tau_1 \vee \tau_2 := \max\{\tau_1, \tau_2\}$, $\tau_1 \wedge \tau_2 := \min\{\tau_1, \tau_2\}$ are stopping times,
3. $\mathcal{F}_\tau$ is a σ-algebra,
4. If $\tau_1 \leq \tau_2$, then $\mathcal{F}_{\tau_1} \subset \mathcal{F}_{\tau_2}$,
5. $\mathcal{F}_{\tau_1 \wedge \tau_2} = \mathcal{F}_{\tau_1} \cap \mathcal{F}_{\tau_2}$,
6. If $F \in \mathcal{F}_{\tau_1 \vee \tau_2}$, then $F \cap \{\tau_2 \leq \tau_1\} \in \mathcal{F}_{\tau_1}$,
7. $\mathcal{F}_{\tau_2 \vee \tau_1} = \sigma(\mathcal{F}_{\tau_1}, \mathcal{F}_{\tau_2})$,
8. If X is an adapted process, then X_τ is $\mathcal{F}_\tau$-measurable.

Exercise 4 *[5 Pts.]*

Let $(Y_n)_{n\in\mathbb{N}}$ be a Markov chain on the finite state space $S = \{1, \dots, m\}$ with transition matrix $P = \{p_{ij}\}$, i.e. $\mathbb{P}(Y_{n+1} = j | Y_n = i) = p_{ij}$. Let $x = (x(j))_{j=1,\dots,m}$ be the eigenvector of the transition matrix, i.e. there exists some $\lambda \in \mathbb{R}$ such that for all i:

$$\sum_{j=1}^{m} p_{ij} x(j) = \lambda x(i).$$

Let $Z_n = \lambda^{-n} x(Y_n)$. Show that $(Z_n)_{n\in\mathbb{N}}$ is a martingale with respect to the filtration $\mathcal{F}_n := \sigma(Y_1, \dots, Y_n)$.