Institute for Applied Mathematics
WS 2022/23
Prof. Dr. Anton Bovier,
Florian Kreten

Sheet 9, "Introduction to Stochastic Analysis"

Due before December 16, 2022

Remark on stochastic calculus:
Regarding the cross-variation $[X, Y]$ of two processes X and Y, you may use that

$$d[X, Y] = 0 \text{ if either } X \text{ or } Y \text{ is of bounded variation.}$$

Think about, why this works!

Exercise 1 *[3 Pt]*

Show that there exists a local martingale, which is not a martingale.

Exercise 2 *[6 Pt]*

Let W be a standard Brownian motion and let the process Γ be the solution of

$$\Gamma_0 = 1, \quad d\Gamma_t = \Gamma_t \left(\beta_t dt + \gamma_t dW_t\right).$$

Here β and γ are bounded, adapted processes. Assume that there is a $c > 0$ such that $\gamma_s > c$ for all s. Finally, let $T \in (0, \infty)$.

(a) Show that $\Gamma_t \exp\left(-\int_0^t \beta_s ds\right)$ is a local martingale.

(b) Find a probability measure $\mathbb{Q}_T$ s.t. $(\Gamma_t)_{t\leq T}$ is a local martingale under $\mathbb{Q}_T$.

(c) Compute $d\Gamma_t^{-1}$.

(d) Find a probability measure $\mathbb{R}_T$ s.t. $(\Gamma_t^{-1})_{t\leq T}$ is a local martingale under $\mathbb{R}_T$.

Exercise 3 *[5 Pt]*

Let B be standard Brownian motion on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and $\mathbb{P}^b$ a measure defined on $\mathcal{F}_T$ through

$$d\mathbb{P}^b = \exp\left(-b\int_0^T B_s dB_s - \frac{b^2}{2}\int_0^T B_s^2 ds\right) d\mathbb{P}.$$

You may assume that $\mathbb{P}^b$ is a probability measure (why is this true?).

(a) Show that the process

$$W_t = B_t + b\int_0^t B_s ds, \quad 0 \le t \le T$$

is a $\mathbb{P}^b$-Brownian motion.

(b) Show that

$$\int_0^t B_s dB_s = \frac{1}{2}(B_t^2 - t),$$

$\mathbb{P}^b$-almost surely.

(c) Show that for all $t \le T$:

$$\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\alpha B_t^2 - \frac{b^2}{2}\int_0^t B_s^2 ds\right)\right] = \mathbb{E}_{\mathbb{P}^b}\left[\exp\left(-\alpha B_t^2 + \frac{b}{2}(B_t^2 - t)\right)\right].$$

Exercise 4 *[6 Pt]*

Let B be a standard Brownian motion.

(a) Let f be a continuous function on $[0, 1]$ and consider

$$Z := \int_0^1 f(s) dB_s.$$

Show that Z is a Gaussian random variable and compute the variance.

(b) Let $m \in N$ and define the following stochastic integrals:

$$A_m = \sqrt{2}\int_0^1 \cos(2\pi m t) dB_t, \quad B_m = \sqrt{2}\int_0^1 \sin(2\pi m t) dB_t.$$

Show that the following holds true for any $m \ge 1$:

(i) $A_m, B_m \sim \mathcal{N}(0, 1)$.

(ii) A_m and B_m are uncorrelated.