Institute for Applied Mathematics
WS 2022/23
Prof. Dr. Anton Bovier,
Florian Kreten

Sheet 8, "Introduction to Stochastic Analysis"

Due before December 9, 2022

Exercise 1 *[6 Pt]*

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a standard filtered probability space, B_t a one-dimensional Brownian motion, and σ_t an adapted process such that for all finite $t \geq 0$: $\mathbb{E}[\int_0^t \sigma_s^2 \mathrm{d}s] < \infty$. Define the process

$$Y_t = \exp\left(\int_0^t \sigma_s \mathrm{d}B_s - \frac{1}{2} \int_0^t \sigma_s^2 \mathrm{d}s \right). \tag{1}$$

(a) Use Ito's formula to show that

$$dY_t = \sigma_t Y_t dB_t$$

(b) Prove that $(Y_t)_{t\geq 0}$ is a supermartingale.

(c) If σ_t is constant, i.e. $\sigma_t = \sigma$, prove that Y_t is a martingale.

Exercise 2 *[7 Pt]*

Let B be a d-dimensional Brownian motion starting at $x \neq 0$. For $a > 0$ define the stopping time $T_a = \inf\{t \geq 0 : \ |B_t| = a\}$.

1. Let $d = 2$ and $0 < r < |x| < R$. Show that $\log(|B_{t\wedge T_r \wedge T_R}|)$ is a bounded martingale and prove that

   $$\mathbb{P}\left[T_r < T_R\right] = \frac{\log R - \log|x|}{\log R - \log r}.$$

   Deduce that B never hits the origin a.s.

2. Let $d = 3$. Show that $|B_{t\wedge T_r \wedge T_R}|^{-1}$ is a bounded martingale and that

   $$\mathbb{P}\left[T_r < T_R\right] = \frac{R^{-1} - |x|^{-1}}{R^{-1} - r^{-1}}.$$

   Deduce that $\mathbb{P}[T_r < \infty] = r|x|^{-1}$.

Exercise 3 [*7 Pt*]

1. Let M be a continuous local martingale. Show that for all $a < b$ and on a set of probability one,

$$[M]_a(\omega) = [M]_b(\omega) \Leftrightarrow \forall_{t\in[a,b]} : M_t(\omega) = M_a(\omega).$$

2. Consider two independent, continuous martingales M, N. Show that $[M, N] = 0$.
Hint: If $(\mathfrak{F}_t)_t$ and $(\mathfrak{G}_t)_t$ are the canonical filtrations of M and N respectively, then

$$\sigma(\mathfrak{F}_t \cup \mathfrak{G}_t) = \sigma\left(\left\{A_1 \cap A_2 \middle| A_1 \in \mathfrak{F}_t, A_2 \in \mathfrak{G}_t\right\}\right).$$

Remarks: (1) This implies that $[B^i, B^j] = \delta_{ij}t$, if $B = (B^1, \ldots, B^d)$ is a $d-$dimensional Brownian motion. (2) The statement holds also for local martingales.