Institute for Applied Mathematics
WS 2022/23
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Sheet 7, "Introduction to Stochastic Analysis"

Due before December 02, 2022

Exercise 1 *[4 Pt]*

Show that the following σ-algebras are the same:

- $\sigma(\mathcal{E}_b) = \sigma(\{X : \mathbb{R}_+ \times \Omega \to \mathbb{R} — X \in \mathcal{E}_b \text{ and the map } (t,\omega) \mapsto X_t(\omega) \text{ is measurable}\})$
- $\sigma(\{X : \mathbb{R}_+ \times \Omega \to \mathbb{R} — X \text{ is adapted and left-continuous on } (0,\infty)\})$
- $\sigma(\{X : \mathbb{R}_+ \times \Omega \to \mathbb{R} — X \text{ is adapted and continuous on } [0,\infty)\})$

Exercise 2 *[5 Pt]*

Let B be the a one-dimensional Brownian motion. Let $T \in (0,\infty)$ and let I^n be a sequence of partitions of the interval $[0,T]$, i.e. a sequence of families of points $0 = u_0 < u_1 < \cdots < u_n = T$. For each j, denote by $u_j^* = \frac{1}{2}(u_j + u_{j+1})$ the midpoint of the interval $[u_j, u_{j+1}]$.

(a) Show that

$$\lim_{\|I^n\|\to 0} \sum_{j=0}^{n-1} |B_{u_j^*} - B_{u_j}|^2 = \frac{T}{2} \qquad \text{in probability.}$$

(b) Define the *Stratonovich-integral* of B with respect to B by

$$\int_0^T B_t \circ dB_t := \lim_{\|I^n\|\to 0} \sum_{j=0}^{n-1} B_{u_j^*}(B_{u_{j+1}} - B_{u_j}),$$

where the limit is understood in the sense of convergence in probability. Show that

$$\int_0^T B_t \circ dB_t = \int_0^T B_t \, dB_t + \frac{T}{2}.$$

Exercise 3 *[6 Pt]*

Let M be a continuous local martingale. Show that

1. If $\mathbb{E}(\sup_{s\in[0,t]} |M_s|) < \infty$ for all $t \geq 0$, then M is a martingale.
2. If $\mathbb{E}([M]_t) < \infty$ for all $t \geq 0$, then M is a martingale.
3. If M is non-negative and integrable for all t, then M is a supermartingale.
4. If for all $t \geq 0$, the family $\{M_{t\wedge T} \,|\, T$ is a bounded stopping time$\}$ is uniformly integrable, then M is a martingale.

   Hint: Show first that, if an integrable, adapted and cadlag stochastic process X satisfies $\mathbb{E}(X_0) = \mathbb{E}(X_T)$ for all bounded stopping times T, then X is a martingale.

Exercise 4 *[5 Pt]*

Let $u \in C_b^2(\mathbb{R})$ and let $f \in C_b^2(\mathbb{R}_+ \times \mathbb{R})$ be a solution of

$$\partial_t f = \frac{1}{2}\partial_{xx}^2 f, \;\; f(0,x) = u(x).$$

Under $\mathbb{P}_x$, let B be a one-dimensional Brownian motion starting from x. Show that

$$f(t,x) = \mathbb{E}_x[u(B_t)].$$

Hint: Use Itō's formula to find a suitable martingale.