Institute for Applied Mathematics
WS 2022/23
Prof. Dr. Anton Bovier,
Florian Kreten

Sheet 6, "Introduction to Stochastic Analysis"

Due before November 25, 2022

Exercise 1 *[6 Pt]*

Let $g : \mathbb{R}_+ \mapsto \mathbb{R}$ and $t \in \mathbb{R}_+$.

1. Let g be of bounded variation and let $s \mapsto R_g(s)$ denote the corresponding variation of g, defined in Section 3.1 from the lecture notes. Show that g is right-continuous, if and only if R_g is right-continuous.

   Hint: Show that $|R_g(s) - R_g(s+)| = |g(s) - g(s+)|$.

2. Let g be right-continuous. Show that the following statements are equivalent.

   (a) g is of bounded variation on $[0, t]$,

   (b) there exist two unique measures μ_{g_1} and μ_{g_2} on $\mathbb{R}_+$ such that for all $0 \leq r \leq s \leq t$

   $$\mu_{g_1}(\,(r, s]\,) - \mu_{g_2}(\,(r, s]\,) = g(s) - g(r), \qquad \text{and}$$

   $$\mu_{g_1}(\,(0, s]\,), \mu_{g_2}(\,[0, s]\,) < \infty.$$

3. Let g be right-continuous and $f : \mathbb{R}_+ \mapsto \mathbb{R}$ be left-continuous and locally bounded. Following the notation from Section 3.1 from the lecture notes, show that for any $t \in [0, \infty)$:

   $$\int_0^t f d\mu_{g_1} - \int_0^t f d\mu_{g_2} = \lim_{m \uparrow \infty} \sum_{I^{(m)}} f dg =: \int_0^t f dg.$$

Exercise 2 *[4 Pt]*

Let $T, y > 1$. Compute the following integrals explicitely:

i) $\int_0^T \sin(x) d\cos(x) + \int_0^T \cos(x) d\sin(x)$

ii) $\int_0^T 1_{[1,y)}(x) d|x - 2|$

iii) $\int_0^T |x - 2| d1_{[1,y)}(x)$

You may use that the Stieltjes integral coincides with the usual Riemann integral as soon as the integrator is a smooth function; in other words, for $g \in C^1(\mathbb{R}_+)$ and a locally bounded, Borel-measurable function $f : \mathbb{R}_+ \to \mathbb{R}$ it holds:

$$\int_0^t f(s)dg(s) = \int_0^t f(s)g'(s)ds.$$

Exercise 3 *[5 Pt]*

Let $g : [0,1] \to \mathbb{R}$ be right-continuous and I^n a sequence of partitions of the interval $[0,1]$, i.e. a sequence of families of points $0 = u_0^n < u_1^n < \cdots < u_n^n = 1$ such that $\lim_{n\to\infty} \|I^n\| = 0$, where $\|I^n\| = \max_{k=1,\dots,n}\left(u_k^n - u_{k-1}^n\right)$. For any continuous $f \in C([0,1])$ define the sum

$$S_n(f) := \sum_{k:t_k,t_{k+1}\in I^n} f(t_k)\Big(g(t_{k+1}) - g(t_k)\Big),$$

and assume that the limit $\lim_{n\to\infty} S_n(f)$ exists and is finite for all $f \in C([0,1])$. Show that g is necessarily of finite variation.
Use the following statement, which is called Banach-Steinhaus Theorem: Let X be a Banach space, Y a normed vector space and $\{T_i\}, i \in I$ be a family of bounded linear operators $T_i : X \to Y$. If $\sup_i ||T_i x|| < \infty$ for all $x \in X$, then even $\sup_i ||T_i|| < \infty$.

Exercise 4 *[5 Pt]*

Let M be a continuous martingale. Show that, if $\mathbb{P}(\sup_{t\geq 0} [M]_t < \infty) = 1$, then $\lim_{t\to\infty} M_t$ exists almost surely.