Institute for Applied Mathematics
WS 2022/23
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Sheet 5, "Introduction to Stochastic Analysis"

Due before November 18, 2022

Exercise 1 *[5 Pt]*

Let T and S be $\mathfrak{G}_{t+}$-stopping times.

a) If $T > 0$ and $S > 0$ prove that $T + S$ is a $\mathfrak{G}_t$-stopping time.

b) If $T > 0$ and T is a $\mathfrak{G}_t$-stopping time, prove that $T + S$ is a $\mathfrak{G}_t$-stopping time.

Exercise 2 *[5 Pt]*

Let $(N_t)_{t\in\mathbb{R}_+}$ be the Poisson counting process with intensity $\lambda > 0$. For $a \in \mathbb{N}$, define the stopping time $T_a = \inf\{t \in \mathbb{R}_+ : N_t = a\}$. Show that

a) T_a is almost surely finite.

b) $\mathbb{E}[T_a] = \frac{a}{\lambda}$.

c) $\mathrm{Var}[T_a] = \frac{a}{\lambda^2}$.

Hint: Find suitable martingales for proving b) and c).

Exercise 3 *[5 Pt]*

Let $(B_t)_{t\in\mathbb{R}_+}$ be one-dimensional Brownian motion.

i) Let T be a stopping time s.t. $\mathbb{E}(T) < \infty$. Show that:

  a) $\mathbb{E}(B_T) = 0$.

  b) $\mathbb{E}(|B_T|^2) = \mathbb{E}(T)$.

ii) Let $r > 0$, $x \in \mathbb{R}$ and consider the stopping time $T_{x,r} := \inf\{t \in \mathbb{R}_+ : |B_t - x| \geq r\}$. It holds that $\mathbb{E}(T_{x,r}) < \infty$. Show that

$$\mathbb{E}(T_{x,r}) = \begin{cases} r^2 - |x|^2 & |x| < r, \\ 0 & \text{otherwise.} \end{cases}$$

Remark: $\mathbb{E}(T_{x,r}) < \infty$ can be shown easily using Lemma 4.35 in the lecture notes of Stochastic Processes.

Exercise 4 *[5 Pt]*

Let $(B_t)_{t\geq 0}$ be a one-dimensional Brownian motion. For $a, b > 0$ and $x \in \mathbb{R}$ we define the stopping times

$$T_x = \inf\{t \in \mathbb{R}_+ : B_t = x\}, \quad T_{a,b} = T_{-a} \wedge T_b.$$

a) Let $\theta \geq 0$ and $X_t^{\theta,a} = \sinh(\theta(B_t + a)) \exp\left(-\frac{\theta^2}{2}t\right)$. Show that $(X_t^{\theta,a})$ is a martingale.

b) Let $\lambda \geq 0$. Show that

$$\mathbb{E}\left[\exp\left(-\lambda T_{a,b}\right)\right] = \frac{\cosh\left(\frac{a-b}{2}\sqrt{2\lambda}\right)}{\cosh\left(\frac{a+b}{2}\sqrt{2\lambda}\right)}.$$

Hint: Consider $X_t^{\theta,a}$ and $X_t^{\theta,-b}$ and use optional sampling. The following identities might be useful:

$$\sinh(x) + \sinh(y) = 2\sinh\left(\frac{x+y}{2}\right)\cosh\left(\frac{x-y}{2}\right),$$
$$\sinh(2x) = 2\sinh(x)\cosh(x).$$