Institute for Applied Mathematics
WS 2022/23
Prof. Dr. Anton Bovier,
Florian Kreten

Sheet 4, "Introduction to Stochastic Analysis"

Due before November 11, 2022

---

1. (Where is the randomness?) *[5 Pt]*

Let $f : [0,1] \to \mathbb{R}$ be a Lipschitz function, namely $|f(x) - f(y)| \leq K|x-y|$ for some $K > 0$, and $x, y \in [0,1]$. Let furthermore f_n be the function obtained from f through linear interpolation at the points $\{k2^{-n}\}_{0 \leq k \leq 2^n}$. Define

$$M_n(x) = \begin{cases} f_n'(x) & x \neq k2^{-n}, 0 \leq k \leq 2^n \\ \lim_{(x_m)\downarrow x} f_n'(x_m), & x = k2^{-n}, 0 \leq k \leq 2^n - 1 \\ f(1), & x = 1 \end{cases}$$

1. Show that $(M_n)_{n\in\mathbb{N}}$ is a martingale (for some, to be given, suitable filtered probability space).
2. Use 1. to show that there exists a measurable, bounded function $g : [0,1] \to \mathbb{R}$ such that for all $x \in [0,1]$

$$f(x) = f(0) + \int_0^x g(y)dy.$$

   Hint: Use Theorem 1.27 from the lecture notes.

Exercise 2 *[5 Pt]*

Let B be a one-dimensional, continuous process with $B(0) = 0$. Assume that the process

$$M_t^\alpha := \exp\left(\alpha B_t - \frac{\alpha^2}{2}t\right)$$

is a martingale w.r.t. $(\mathcal{F}_t)_{t\geq 0}$ for each $\alpha \in \mathbb{R}$. Show that B is a $\mathcal{F}_t-$Brownian motion.

Hint: Let $n \in \mathbb{N}$ and let $X_1, \ldots, X_n$ be random variables. Then the distribution of $X = (X_1, \ldots, X_n)$ is completely characterized by its Laplace transform $\mathbb{E}[e^{\langle\lambda,X\rangle}]$ (if it exists), where $\lambda \in \mathbb{R}^n$.

Exercise 3 *[10 Pt]*

Let $(\Omega, \mathfrak{G} : \mathbb{P}, (\mathfrak{G}_t, t \in \mathbb{R}_+))$ be a filtered space. Let S, T be $\mathfrak{G}_t$-stopping times. Show that:

1. $T + \theta$ is a $\mathfrak{G}_t$-stopping time, where $\theta > 0$.
2. $T + S$ is a $\mathfrak{G}_t$-stopping time.
3. $T \wedge S = \min\{T, S\}$ and $T \vee S = \max\{T, S\}$ are $\mathfrak{G}_t$-stopping times.
4. If $F \in \mathfrak{G}_S$, then $F \cap \{S \leq T\} \in \mathfrak{G}_T$.
5. If $S \leq T$ on Ω, then $\mathfrak{G}_S \subset \mathfrak{G}_T$.
6. $\mathfrak{G}_{T \wedge S} = \mathfrak{G}_T \cap \mathfrak{G}_S$.
7. $\{\{T < S\}, \{T \leq S\}, \{T = S\}\} \subset \mathfrak{G}_T \cap \mathfrak{G}_S$.