Institute for Applied Mathematics
WS 2022/23
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Sheet 3, "Introduction to Stochastic Analysis"

Due before November 04, 2022

Exercise 1 *[5 Pt]*

Let $(X_t)_{t\geq 0}$ be a non-negative, right-continuous submartingale w.r.t to the filtration $(\mathcal{F}_t)_{t\geq 0}$. Show that the following statements are equivalent:

1. The family $(X_t, t \in \mathbb{R}_+)$ is uniformly integrable.
2. The L_1-limit $\lim_{t\to\infty} X_t$ exists.
3. There exists an integrable r.v. X_∞ such that
   - $\lim_{t\to\infty} X_t = X_\infty$ a.s.
   - $(X_t, \mathcal{F}_t)_{t\in\mathbb{R}_+\cup\{+\infty\}}$ is a submartingale, where $\mathcal{F}_\infty := \sigma\left(\cup_{t\in\mathbb{R}_+}\mathcal{F}_t\right)$

Exercise 2 *[4 Pt]*

Let Z be a square integrable random variable on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, \mathbb{P})$. We define the stochastic process $(M_t)_{t\in\mathbb{R}_+}$ by

$$M_t = \mathbb{E}[\, Z \mid \mathcal{F}_t\,].$$

Prove that:

1. $(M_t)_{t\in\mathbb{R}_+}$is a martingale.
2. $\mathbb{E}[\sup_{t\in\mathbb{R}_+} M_t^2\,] \leq 4\mathbb{E}[\,Z^2\,]$.

Exercise 3 *[6 Pt]*

Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, \mathbb{P})$ and $1 < p < \infty$. For a process $(M_t)_{t\in\mathbb{R}_+}$ we define $\|M\| = (\sup_{t\in\mathbb{R}_+}\mathbb{E}[|M_t|^p])^{1/p}$. Furthermore, consider the space

$$\mathcal{M}^p = \left\{(M_t)_{t\in\mathbb{R}_+} \text{ is a martingale, } M_0 = 0, \|M\| < \infty\right\}.$$

1. Show that $(\mathcal{M}^p, \|\cdot\|)$ is a Banach space.
   Hint: Consider the random variable $M_\infty = \lim_{t\to\infty} M_t$.

2. Show that the map

$$\boldsymbol{j} : \mathcal{M}^p \to L^p(\Omega, \mathcal{F}, \mathbb{P})$$
$$(M_t)_{t \in \mathbb{R}_+} \mapsto M_\infty$$

is an isometry between Banach spaces.

Exercise 4 *[5 Pt]*

Fix $t > 0$. Let for all $n \in \mathbb{N}$, $\Pi^n := \{0 = t_0 < t_1 < \cdots < t_n = t\}$ be a sequence of partitions of $[0, t]$ such that $\lim_{n\to\infty} \|\Pi^n\| = 0$, where $\|\Pi^n\| := \max_{1 \le k \le n} |t_k - t_{k-1}|$. For $q > 0$ consider the q-*Variation* of a process X w.r.t. Π^n defined as

$$V_t^{(q)}(\Pi^n) := \sum_{k=1}^{n} |X_{t_k} - X_{t_{k-1}}|^q.$$

Assume that X is a continuous, adapted process s.t. for some $p > 0$

$$\lim_{n\to\infty} V_t^{(p)}(\Pi^n) = L_t^p \text{ a.s.},$$

for a random variable L_t^p, which takes values on $[0, \infty]$ a.s.

1. Let $0 < q < p$ and assume that $L_t^p > 0$ a.s. Show that $\lim_{n\to\infty} V_t^{(q)}(\Pi^n) = \infty$ a.s.
2. Let $q > p$ and assume that $L_t^p < \infty$ a.s. Show that $\lim_{n\to\infty} V_t^{(q)}(\Pi^n) = 0$ a.s.