Institute for Applied Mathematics
WS 2022/23
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Sheet 2, "Introduction to Stochastic Analysis"

Due before October 28, 2022

Exercise 1 *[4 Pt]*

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, \mathbb{P})$ be a filtered probability space.

1. Let X be a martingale and ϕ a convex function satisfying $\mathbb{E}(|\phi(X_t)|) < \infty$ for all $t \in \mathbb{R}_+$. Show that $\big(\phi(X_t)\big)_{t\in\mathbb{R}_+}$ is a submartingale.

2. Let X be a submartinale and ϕ a convex non-decreasing function with the property $\mathbb{E}(|\phi(X_t)|) < \infty$ for all $t \in \mathbb{R}_+$. Show that $\big(\phi(X_t)\big)_{t\in\mathbb{R}_t}$ is also a submartingale.

Exercise 2 *[5 Pt]*

Denote by W a standard one-dimensional Brownian motion, and let Z be a random variable independent of W such that

$$\mathbb{P}(Z = +1) = \mathbb{P}(Z = -1) = \frac{1}{2}.$$

Let furthermore $t_\star \in [0, \infty)$. We define another stochastic process $B = (B_t)_{t\in\mathbb{R}_+}$ through

$$B_t = W_t \mathbb{1}_{\{t<t_\star\}} + (W_{t_\star} + Z(W_t - W_{t_\star}))\, \mathbb{1}_{\{t\geq t_\star\}}$$

First describe this process intuitively. Then show that B is a standard Brownian motion as well.

Exercise 3 *[5 Pt]*

Let $(N_t)_{t\in\mathbb{R}_+}$ be the Poisson counting process with intensity $\lambda > 0$, i.e a Poisson point process with intensity measure $\lambda \cdot \mathcal{L}(\mathbb{R}_+)$. Here, $\mathcal{L}(\mathbb{R}_+)$ is the Lebesgue measure on $\mathbb{R}_+$. Show that

1. The process $(N_t - \lambda t)_{t\in\mathbb{R}_+}$ is a martingale.

2. The process $((N_t - \lambda t)^2 - \lambda t)_{t\in\mathbb{R}_+}$ is a martingale.

Exercise 4 *[6 Pt]*

Let X be the compound Poisson process, i.e.

$$X_t \equiv \sum_{i=1}^{N_t} Y_i,$$

with $(N_t)_{t\in\mathbb{R}_+}$ as in exercise 3, and $Y_i, i \in \mathbb{N}$ are iid real random variables that are independent of $(N_t)_{t\in\mathbb{R}_+}$. Assume $\mathbb{E}[|Y_1|] < \infty$ and $\mathbb{E}[\exp(Y_1)] < \infty$.

1. Find $c \in \mathbb{R}$ such that the process $(X_t - ct)_{t\in\mathbb{R}_+}$ is a martingale.
2. Find $a \in \mathbb{R}$ such that the process $(\exp(X_t - at))_{t\in\mathbb{R}_+}$ is a martingale.