Institute for Applied Mathematics
WS 2022/23
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Sheet 12, “Introduction to Stochastic Analysis”

Due before January 20, 2022

Definition. A (only for simplicity one-dimensional) Gaussian process $X = (X_t)_{t\in[0,1]}$ with continuous paths is called a *Brownian bridge* (from 0 to 0), if

- $\mathbb{E}[X_t] = 0$ for all $t \in [0,1]$,
- $\mathrm{Cov}[X_s, X_t] = s(1-t)$ for all $0 \le s \le t \le 1$.

Exercise 1 *[0 Pt]*

Let B be a one-dimensional Brownian motion.

1. Define $X_t = B_t - tB_1$ for all $t \in [0,1]$. Show that X is a Brownian bridge and that X is independent of B_1.

   Hint: Linear transformations of Gaussian vectors are Gaussian.

2. Define $X_t = (1-t)B_{\frac{t}{1-t}}$ for all $t \in [0,1)$, and $X_1 = 0$. Show that X is a Brownian bridge.
   Hint: Law of iterated logarithm.

3. Let X be a Brownian bridge. Show that X is the solution of the following SDE:

   $$dX_t = -\frac{X_t}{1-t}dt + dW_t, t \in (0,1),$$

   with $X_0 = 0$, where W is a Brownian motion.
   Hint: Time-changed Brownian motion (see sheet 8,9).

Exercise 2 *[0 Pt]*

Let $(B_t)_{t\ge 0}$ be a standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$ and set for all $t \ge 0$:

$$H_t = \exp\left(B_t - \frac{1}{2}t\right).$$

Use Girsanov's theorem to compute $\mathbb{E}^{\mathbb{P}}[H_t \log H_t]$ for any $t \in \mathbb{R}_+$.

Exercise 3 *[0 Pt]*

Let $(B_t)_{t\geq 0}$ be a one-dimensional Brownian motion. In this exercise, we compute pathwise solutions of SDEs of the form

$$dX_t = f(t, X_t)dt + c(t)X_t dB_t, \qquad X_0 = x > 0, \tag{1}$$

where $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ and $c : \mathbb{R}^+ \to \mathbb{R}$ are deterministic, continuous functions. In order to solve (1), we proceed as in the *variation of constants* method, which is used to solve ODEs:

1. We know a solution Z_t of the equation with $f \equiv 0$:

$$Z_t = x \cdot e^{\int_0^t c(s)dB_s - \frac{1}{2}\int_0^t c(s)^2 ds}. \tag{2}$$

2. Now for solving the equation in the general case, use the Ansatz

$$X_t \;=\; C_t \cdot Z_t,$$

where C_t is a process of finite variation.

Show that the SDE satisfied by $(C_t)_{t\geq 0}$ has the form

$$\frac{dC_t(\omega)}{dt} \;=\; \frac{f(t, Z_t(\omega)\cdot C_t(\omega))}{Z_t(\omega)}, \;\; C_0 = 1. \tag{3}$$

Hint: For each $\omega \in \Omega$, this is a *deterministic* differential equation for the function $t \mapsto C_t(\omega)$. We can therefore solve (3) pathwise, with ω as a parameter to find $C_t(\omega)$.

3. Let α be a constant. Apply the previous method to solve the following SDE:

$$dX_t = \frac{1}{X_t}\,dt + \alpha X_t\,dB_t, \qquad X_0 = x > 0.$$

The following exercise will not be discussed in the tutorial classes:

Exercise 4 *[0 Pt]*

Let B_t be a 1-dimensional Brownian motion. Give an explicit solution (with proof) of the following SDE:

$$dX_t = \frac{1}{2}X_t dt + \sqrt{1 + X_t^2}\,dB_t.$$