Institute for Applied Mathematics
WS 2022/23
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Sheet 11, "Introduction to Stochastic Analysis"

Due before January 13, 2023

This is the last sheet that is relevant for your admission to the exam. We will soon start the assignment for the oral exams in February, regularly check the *homepage of the lecture*.

Exercise 1 *[10 Pt]*

Let B_t be a three-dimensional Brownian motion, starting at $x \neq 0$ and let $X_t := ||B_t||^{-1}$. Prove the following statements:

i) $(X_t)_{t\geq 0}$ is a local martingale.

ii) $\mathbb{E}[\sup_{t\geq 0} X_t] = \infty$.
Hint: $\mathbb{E}[Z] = \int_0^\infty \mathbb{P}[Z \geq y]dy$ for $Z \geq 0$.

iii) The family $(X_t)_{t\geq 0}$ is uniformly integrable.
Hint: You may use the inequality

$$\sup_{t\geq 0} \mathbb{E}\left[X_t^p\right] \leq c \int_{\mathcal{B}} \frac{1}{|y|^p} dy + 1\,,$$

which holds for all $p > 0$ and the unit ball $\mathcal{B} = \{z \in \mathbb{R}^3 : \ |z| < 1\}$.

iv) $(X_t)_{t\geq 0}$ is not a martingale. Towards this goal, prove that

$$\mathbb{E}[X_t^p] \leq \frac{1}{2t^{p/2}} \int_0^\infty y^{-p/2} e^{-y/2} dy.$$

Hint: Chi-squared distribution!

v) Conclude that $\lim_{t\to\infty} ||B_t|| = \infty$ almost surely.

Exercise 2 *[3 Pt]*

Let B_t be a one-dimensional Brownian motion. Find the SDEs satisfied by the following processes:

1. $X_t = B_t/(1+t)$ for all $t \geq 0$,

2. $X_t = \sin(B_t)$ for all $t \geq 0$,
3. $(X_t, Y_t) = (a\cos(B_t), b\sin(B_t))$ for all $t \geq 0$, where $a, b \in \mathbb{R}$ with $ab \neq 0$.

Exercise 3 *[4 Pt]*

Let B_t be a one-dimensional Brownian motion and $\xi \in \mathbb{R}$. Furthermore, let $A(t), a(t)$ and $\sigma(t)$ be deterministic, measurable and locally bounded. Show that the SDE

$$dX_t = [A(t)X_t + a(t)]dt + \sigma(t)dB_t,\ X_0 = \xi,$$

admits the unique solution

$$X_t = \phi(t)\left(\xi + \int_0^t \phi^{-1}(s)a(s)ds + \int_0^t \phi^{-1}(s)\sigma(s)dB_s\right),\ 0 \leq t < \infty,$$

where ϕ is the unique solution of the ODE

$$\frac{d}{dt}\phi(t) = A(t)\phi(t),\ \phi(0) = 1.$$

Exercise 4 *[3 Pt]*

Suppose S_t is a solution of the SDE $dS_t = S_t(b_t dt + \sigma_t dB_t)$ (modeling a stock), where B_t is a one-dimensional Brownian motion. Let b_t, σ_t and r_t be deterministic, locally bounded functions. Further, assume that $\sigma > c$ for some $c > 0$.

1. Let $T > 0$ be finite. Show that there is a probability measure $\mathbb{Q}_T$ such that for $t < T$ and under $\mathbb{Q}_T$:
$$dS_t = S_t(r_t dt + \sigma_t d\widetilde{B}_t),$$
where $\widetilde{B}$ is a $\mathbb{Q}_T$-Brownian motion (up to time T).
2. Set $R_t := \exp\left(-\int_0^t r(s)ds\right)$. Show that $R_t S_t$ is a martingale under $\mathbb{Q}_T$.

The following exercise is optional and will neither be graded, nor discussed in the tutorial classes:

Exercise 5 *[0 Pt]*

Let $\mathbb{P}$ and $\mathbb{Q}$ be probability measures on $(\Omega, \mathcal{A})$ such that $\mathbb{Q}$ is absolutely continuous w.r.t. $\mathbb{P}$ (for all $A \in \mathcal{A}$: if $\mathbb{P}(A) = 0$ then also $\mathbb{Q}(A) = 0$). We will show a version of the famous **Radon-Nikodym theorem** using the martingale theory we have developed so far. The goal of this exercise is to prove the existence of a relative density under the assumption of absolute continuity. A relative density of $\mathbb{Q}$ w.r.t $\mathbb{P}$ is a measurable random variable $Z : \Omega \to [0, \infty)$, such that for all $A \in \mathcal{A}$:

$$\mathbb{Q}(A) = \int_A Z(\omega) d\mathbb{P}(\omega).$$

Let $\mathcal{A} = \sigma(\cup_n \mathcal{F}_n)$, where $(\mathcal{F}_n)_{n \in \mathbb{N}}$ is a filtration of $\mathcal{A}$ consisting of finite σ-algebras $\mathcal{F}_n$, i.e. $\mathcal{F}_n = \sigma(B_{n,1}, \ldots, B_{n,k_n})$ such that $\cup_i B_{n,i} = \Omega$.

1. Write down a relative density Z_n of $\mathbb{Q}$ w.r.t $\mathbb{P}$ on $\mathcal{F}_n$ and show that $(Z_n)_{n \in \mathbb{N}}$ is a non-negative martingale under $\mathbb{P}$.
2. Show that the limit Z_∞ exists $\mathbb{P}$-almost surely and in $L^1(\Omega, \mathcal{A}, \mathbb{P})$.
3. Conclude that Z_∞ is a relative density of $\mathbb{Q}$ w.r.t $\mathbb{P}$ on $\mathcal{A}$.