Institute for Applied Mathematics
WS 2022/23
Prof. Dr. Anton Bovier,
Florian Kreten

Sheet 10, "Introduction to Stochastic Analysis"

Due before January 06, 2023

Exercise 1 *[8 Pt]*

If $c(t) = (x(t), y(t))$ is a smooth curve in $\mathbb{R}^2$ with $c(0) = (0, 0)$, then

$$A(t) = \int_0^t [x(s)y'(s) - y(s)x'(s)]\, ds = \int_0^t x dy - \int_0^t y dx$$

describes the area that is covered by the secant from the origin to $c(s)$ in the interval $[0, t]$. Analogously, for a 2-dim BM $B_t = (X_t, Y_t)$ with $B_0 = 0$, one defines the *Lévy area* as

$$A_t = \int_0^t X_s dY_s - \int_0^t Y_s dX_s.$$

a) Let $\alpha(t), \beta(t)$ be C^1-functions, $p \in \mathbb{R}$ and

$$V_t = ipA_t - \frac{\alpha(t)}{2}(X_t^2 + Y_t^2) + \beta(t).$$

Show that e^{V_t} is a local martingale provided that $\alpha'(t) = \alpha(t)^2 - p^2$ and $\beta'(t) = \alpha(t)$.

b) Let $t_0 \in [0, \infty)$. The solutions of the ODE for α and β with $\alpha(t_0) = \beta(t_0) = 0$ are

$$\alpha(t) = p \cdot \tanh(p \cdot (t_0 - t)), \quad \beta(t) = -\log\cosh(p \cdot (t_0 - t)).$$

Conclude that

$$\mathbb{E}\left[\exp\left(ipA_{t_0}\right)\right] = \frac{1}{\cosh(pt_0)}, \quad \forall p \in \mathbb{R}.$$

Remark: This shows that the distribution of A_t is absolutely continuous with density

$$f(x) = \frac{1}{2t\cosh\left(\frac{\pi x}{2t}\right)}.$$

Exercise 2 *[6 Pt]*

Let $u, \alpha : \mathbb{R} \to \mathbb{R}$ be bounded and twice continuously differentiable and let $f \in C^2(\mathbb{R}_+ \times \mathbb{R})$ be a bounded solution of

$$\partial_t f = \alpha \partial_x f + \frac{1}{2}\partial^2_{xx} f, \qquad f(0,x) = u(x).$$

Show that

$$f(t,x) = \mathbb{E}_x \left[\exp\left(\int_0^t \alpha(s) dB_s - \frac{1}{2} \int_0^t \alpha(s)^2 ds \right) u(B_t) \right],$$

where, under $\mathbb{P}_x$, B_t is a Brownian motion starting from x.

Exercise 3 *[6 Pt]*

Let B_t be a d-dimensional Brownian motion and let $(X_t) = (X_t^1, \dots, X_t^d)$ be a solution of the SDE

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t \quad t \geq 0, \quad X_0 = x \in \mathbb{R}^d.$$

We assume (for simplicity) that $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ are bounded and Lipschitz continuous. Determine the limits

$$\lim_{t\to 0} \frac{1}{t}\mathbb{E}[X_t^i - x^i],$$

and

$$\lim_{t\to 0} \frac{1}{t}\mathbb{E}[(X_t^i - x^i)(X_t^j - x^j)].$$

Remarks:

1. $\sigma(t)dB_t = \left(\sum_{i=1}^n \sigma_{ji}(t) dB_t^i \right)_{j=1,\dots,n}$
2. This is the reason why b is called drift vector and $\sigma\sigma^T$ is called diffusion matrix.

The stochastic analysis team wishes you a merry Christmas and a good start into the new year!

The following exercise is optional and will not be graded:

Exercise 4 *[0 Pt]*

Let B be a one-dimensional Brownian motion and $h \in L^2([0,1],\lambda)$ be a deterministic function. Consider the Itō-integral

$$I_t := \int_0^t h(s)dB_s \qquad \text{for } 0 \leq t \leq 1.$$

1. Show that I_t is normally distributed with mean zero and variance

$$\tau(t) = \int_0^t h(r)^2 dr.$$

2. Prove that the increments of $(I_t)_{t\in[0,1]}$ are independent with law

$$I_t - I_s \sim N(0, \tau(t) - \tau(s)) \qquad \text{for } 0 \leq s \leq t.$$

3. Conclude that the process $(I_t)_{t\in[0,1]}$ has the same law as the **time-changed Brownian motion** $t \to B_{\tau(t)}$.