Institute for Applied Mathematics
WS 2022/23
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Sheet 1, "Introduction to Stochastic Analysis"

Due before October 21, 2022
Details about the electronic submission in the exercise classes

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Exercise 1 *[5 Pt]*

Consider a random variable X_1 with a $\mathcal{N}(0,1)$ distribution. Let Y be another random variable which is independent of X_1 and for which we have $\mathbb{P}(Y=1) = \frac{1}{2} = \mathbb{P}(Y=-1)$. Further, define $X_2 := Y \cdot X_1$. Clearly, X_2 has a $\mathcal{N}(0,1)$ distribution. Show that the following statements hold:

1. X_1 and X_2 are uncorrelated but not independent.
2. (X_1, X_2) does not have a two-dimensional Gaussian distribution.

Definition

A process $(B_t)_{t\in\mathbb{R}_+}$ that is adapted to a filtration $(\mathcal{F}_t)_{t\geq 0}$ is called (standard) Brownian motion if it has the following properties:

- $B_0 = 0$ a.s.
- B has independent increments: for every $t > s \geq 0$, $W_t - W_s$ is independent of $\mathcal{F}_s$
- B has Gaussian increments: for $u \geq 0$, $B_{t+u} - B_t$ is normally distributed with mean 0 and variance u.
- For almost all ω, the path $(B_t)_{t\in\mathbb{R}_+}(\omega)$ is continuous in t (B is a.s. continuous)

Exercise 2 *[3 Pt]*

Show that $(B_t)_{t\in\mathbb{R}_+}$ is the one-dimensional Brownian motion, if and only if $(B_t)_{t\in\mathbb{R}_+}$ is a centered Gaussian process with continuous paths and such that $\mathrm{Cov}(B_t, B_s) = t \wedge s$ for all $s, t \geq 0$.

Hint: take a look at section 3 in the script of **Stochastic Processes**.

Exercise 3 *[6 Pt]*

Let $(B_t)_{t\in\mathbb{R}_+}$ be the Brownian motion. Define the processes $(B_t^{(1)})_{t\in\mathbb{R}_+}, (B_t^{(2)})_{t\in\mathbb{R}_+}, (B_t^{(3)})_{t\in\mathbb{R}_+}$ by

1. $B_t^{(1)} = -B_t$,
2. $B_t^{(2)} = B_{t+r} - B_r$ for some $r > 0$,
3. $B_t^{(3)} = \frac{1}{c} B_{c^2 t}$ for some $c > 0$.

Show that $(B_t^{(1)})_{t\in\mathbb{R}_+}, (B_t^{(2)})_{t\in\mathbb{R}_+}, (B_t^{(3)})_{t\in\mathbb{R}_+}$ are Brownian motions as well.

Exercise 4 *[6 Pt]*

Let $(B_t, t \in \mathbb{R}_+)$ be a (one-dimensional) standard Brownian motion.

1. Let $Z := \sup_{t\geq 0} B_t$. Show that $cZ \overset{(d)}{=} Z$ for all $c > 0$ (i.e., cZ and Z have the same laws). Conclude that the law of Z is concentrated on $\{0, \infty\}$.
2. Show that $\mathbb{P}(Z = 0) \leq \mathbb{P}(B_1 \leq 0)\mathbb{P}(\sup_{t\geq 0}(B_{1+t} - B_1) = 0)$ and conclude that $\mathbb{P}(Z = 0) = 0$.
3. Conclude that $\mathbb{P}(\sup_{t\geq 0} B_t = +\infty, \inf_{t\geq 0} B_t = -\infty) = 1$. In other words, paths of the Brownian motion oscillate a.s. infinitely often between $+\infty$ and $-\infty$.
   Another proof of this fact goes through the so-called "law of the iterated logarithm", but it is way less elementary!