Institute for Applied Mathematics
WS 2022/23
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Sheet 0, "Introduction to Stochastic Analysis"

These exercises will be discussed in the first tutorial. You are encouraged to study them in advance, but do not have to submit any solutions.

If you have any questions regarding the organization of the lecture write an email to florian.kreten@uni-bonn.de.

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1. (Conditional expectation)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\mathcal{G} \subset \mathcal{F}$ a sub-σ-algebra. Let X, Y absolutely integrable random variables. First, define the conditional expectation $\mathbb{E}(X|\mathcal{G})$. Then prove the following statements:

1. The map $X \to \mathbb{E}(X|\mathcal{G})$ is linear.
2. If $\mathcal{B} \subset \mathcal{G}$ is a σ-algebra, then $\mathbb{E}[\mathbb{E}(X|\mathcal{G})|\mathcal{B}] = \mathbb{E}(X|\mathcal{B})$, a.s. This is called the tower-property. In particular, this implies $\mathbb{E}[\mathbb{E}[X|\mathcal{G}]] = \mathbb{E}[X]$.
3. If $X \leq Y$ a.s. , then $\mathbb{E}(X|\mathcal{G}) \leq \mathbb{E}(Y|\mathcal{G})$, a.s.
4. $|\mathbb{E}(X|\mathcal{G})| \leq \mathbb{E}(|X||\mathcal{G})$ a.s.
5. Assume that there exists $n \in \mathbb{N}$ and $A_1, \ldots, A_n \subset \Omega$ such that $\{A_1, \ldots, A_n\}$ are pairwise disjoint, $\mathbb{P}(A_i) > 0$ for all i, $\Omega = \cup_{i=1}^n A_i$ and $\mathcal{G} = \sigma\big(\{A_1, \ldots, A_n\}\big)$. Show that $\mathbb{E}(X|\mathcal{G}) = \sum_{i=1}^n \mathbb{E}[X|A_i]\, \mathbb{1}_{A_i}$ a.s., where $\mathbb{E}[X|A_i] = \mathbb{E}[X \mathbb{1}_{A_i}]\, \mathbb{P}[A_i]^{-1}$.

2. (Inequalities)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra. For X, Y absolutely integrable random variables, prove that:

1. Prove the conditional Markov inequality, i.e. show that, if $f : \mathbb{R}_+ \to \mathbb{R}_+$ is non-decreasing and such that $f(|X|)$ is integrable, then
$$\mathbb{P}[|X| \geq \alpha|\mathcal{G}] \leq \frac{1}{f(\alpha)} \mathbb{E}[f(|X|)|\mathcal{G}] \quad \mathbb{P}\text{-a.s.}$$
2. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a convex function, X and $\phi(X)$ be integrable random variables. Prove the conditional Jensen inequality
$$\phi(\mathbb{E}[X|\mathcal{G}]) \leq \mathbb{E}[\phi(X)|\mathcal{G}].$$

Hint: You can use that for $x, y \in \mathbb{R}$ there exists a measurable function $c : \mathbb{R} \to \mathbb{R}$ such that

$$\phi(x) \geq \phi(y) + c(y)(x - y).$$

3. (Independence)

Let X, Y integrable random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\sigma(X)$ the σ-algebra generated by X (what does this mean)? Let $\mathbb{E}[X|Y] := \mathbb{E}[X|\sigma(Y)]$ and call two σ-algebras $\mathcal{A}, \mathcal{B}$ independent if for all $A \in \mathcal{A}, B \in \mathcal{B} : \mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$. From now on, we call X and Y independent if $\sigma(X)$ and $\sigma(Y)$ are independent. Prove:

1. If Y is constant, then X and Y are independent.

2. If $\sigma(X)$ is independent of $\mathcal{G}$, then $\mathbb{E}[X|\mathcal{G}] = \mathbb{E}[X]$ a.s..

3. If X, Y are independent and in $L^2(\mathbb{P})$, then $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$ (they are uncorrelated).