Lecture 8: Exercises

Exercise 8.1 (Liouville property and weak maximum principle)**.** *Let (b, c) be a graph over (X, m) with formal Laplacian $\mathcal{L} = \mathcal{L}_{b,c,m}$. Show that following are equivalent:*

(i) The graph has the Liouville property, i.e., all non-negative superharmonic functions are constant.

(ii) The graph satisfies the weak maximum principle, i.e., if for every non-constant, bounded above function $u \in \mathcal{F}$ and for every $\gamma < \sup_X u$, we have

$$\sup_{X_\gamma} \mathcal{L}u > 0$$

on the superlevel set $X_\gamma := \{x \in X : u(x) > \gamma\}$.

Exercise 8.2 (ℓ^∞-Liouville property and Omori-Yau maximum principle)**.** *Let (b, c) be a graph over (X, m) with formal Laplacian $\mathcal{L} = \mathcal{L}_{b,c,m}$. Show that following are equivalent:*

(i) The graph has the ℓ^∞-Liouville property, i.e., for every $\alpha > 0$, every bounded non-negative function $u \in \mathcal{F}$ such that $(\mathcal{L} + \alpha)u \leq 0$ is trivial.

(ii) There exists $\alpha > 0$ such that every bounded non-negative function $u \in \mathcal{F}$ with $(\mathcal{L} + \alpha)u \leq 0$ is trivial.

(iii) The graph satisfies the Omori-Yau maximum principle, i.e., if for every $u \in \mathcal{F}$ with $\sup_X u \in (0, \infty)$, and $\gamma < \sup_X u$, we have

$$\sup_{X_\gamma} \mathcal{L}u > 0$$

on the superlevel set $X_\gamma := \{x \in X : u(x) > \gamma\}$.

Exercise 8.3. *(a) Show that every finite (b, c) with $c = 0$ satisfies both the Liouville and the ℓ^∞-Liouville property.*

(b) Find examples of infinite graphs which have and do not have the Liouville property.

(c) Find examples of infinite graphs which have and do not have the ℓ^∞-Liouville property.