Lecture 7: Exercises

Let (b, c) be a graph over the infinite discrete measure space (X, m). Recall that we denote the space of functions of finite energy by $\mathcal{D}$ with associated energy functional $\mathcal{Q}$, and we set $\ell^2 := \ell^2(X, m)$. We define

$$\|\cdot\|_{\mathcal{Q}} \colon \mathcal{D} \cap \ell^2 \to [0, \infty), \quad \|f\|_{\mathcal{Q}} := \left(\mathcal{Q}(f) + \|f\|^2\right)^{1/2},$$

which is a norm.

Define a form $Q^{(N)} \colon D(Q^{(N)}) \times D(Q^{(N)}) \to \mathbb{R}$ with domain $D(Q^{(N)}) := \mathcal{D} \cap \ell^2$, acting via

$$Q^{(N)}(f, g) := \mathcal{Q}(f, g), \qquad f, g \in D(Q^{(N)}),$$

and extended to ℓ^2 on the diagonal via $Q^{(N)}(f) := \mathcal{Q}(f, f), f \in D(Q^{(N)})$, and $Q^{(N)}(f) := \infty, f \in \ell^2 \setminus D(Q^{(N)})$. We call this functional the *maximal form*, or Neumann (boundary) form.

Define a form $Q^{(D)} \colon D(Q^{(D)}) \times D(Q^{(D)}) \to \mathbb{R}$ with domain $D(Q^{(D)}) := \overline{C_c(X)}^{\|\cdot\|_{\mathcal{Q}}}$, acting via

$$Q^{(D)}(f, g) := \mathcal{Q}(f, g), \qquad f, g \in D(Q^{(D)}),$$

and extended to ℓ^2 on the diagonal via $Q^{(D)}(f) := \mathcal{Q}(f, f), f \in D(Q^{(D)})$, and $Q^{(D)}(f) := \infty, f \in \ell^2 \setminus D(Q^{(D)})$. We call this functional the *minimal form*, or Dirichlet boundary form.

Exercise 7.1. *(a) Show that $Q^{(N)}$ and $Q^{(D)}$ are closed forms.*

(b) Under the additional assumptions $c = 0$, $m(X) = 1$, and $\lambda_0 = \inf\{Q^{(D)}(f) : f \in D(Q^{(D)}), \|f\| = 1\} > 0$, show that $Q^{(D)} \neq Q^{(N)}$.

(c) Under the additional assumption that the weighted degree

$$\mathrm{Deg}_c \colon X \to [0, \infty), \qquad \mathrm{Deg}_c(x) := \frac{1}{m(x)} \left(\sum_{y \in X} b(x, y) + c(x) \right)$$

is bounded, show that $Q^{(D)} = Q^{(N)}$.

Exercise 7.2. *Let $p \in [1, \infty]$.*

(a) Show the equivalence of the following statements:

(i) $C_c(X)$ is dense in $\ell^p(X, m)$,
(ii) $p \in [1, \infty)$.

(b) Show the equivalence of the following statements for $p \in [1, \infty)$:

(i) $\ell^p(X, m) \subseteq \ell^\infty(X)$,
(ii) $\ell^p(X, m) \subseteq C_0(X) := \overline{C_c(X)}^{\|\cdot\|_{\ell^\infty}}$, i.e., $C_0(X)$ is the space of continuous functions vanishing at infinity,
(iii) There exists $\alpha > 0$ such that $m \geq \alpha$.

(c) Show the equivalence of the following statements for $p \in [1, \infty)$:

(i) $\ell^p(X, m) \supseteq \ell^\infty(X)$,
(ii) $m(X) < \infty$.