Lecture 6: Exercises

Exercise 6.1. *Let T be a positive self-adjoint operator on a Hilbert space H with domain $D(T)$, and let $\varphi\colon [0,\infty) \to [0,\infty)$ be given by $\varphi(t) = t^s$ with $0 < s < 1$. Set $T^s := \varphi(T)$ via functional calculus. Show:*

(a) If $f \in D(T)$ then $f \in D(T^s)$ and

$$\|T^s f\| \leq \|Tf\|^s \, \|f\|^{1-s}.$$

(b) For every $f \in D(T^s)$ the following representation holds:

$$T^s f \;=\; -\frac{s}{\Gamma(1-s)} \int_0^\infty \frac{\left(e^{-tT} - I\right) f}{t^{1+s}} \, dt,$$

where Γ is Euler's Gamma function.

Hints:

1. for (a): Use the full power of the spectral theorem, and then Hölder's inequality.
2. for (b): Integrate
$$-\frac{s}{\Gamma(1-s)} \int_0^\infty \frac{\left(e^{-t\lambda} - 1\right)}{t^{1+s}} \, dt$$
by parts.