Lecture 4: Exercises

Exercise 4.1. *Let $\mathcal{F}$ be the domain of the formal Laplacian $\mathcal{L} = \mathcal{L}_{b,c,m}$ associated with a graph (b, c) over (X, m). Show that the following statements are equivalent:*

(i) *The graph is locally finite.*

(ii) $\mathcal{L}C_c(X) \subseteq C_c(X)$.

(iii) $C(X) = \mathcal{F}$.

Exercise 4.2. *Let $\mathcal{A}\colon C_c(X) \to C_c(X)$ be a symmetric linear operator, i.e.,*

$$\mathcal{A}1_x(y) = \mathcal{A}1_y(x) \quad \text{for all } x, y \in X.$$

Show the following equivalence:

(i) $\mathcal{A} = \mathcal{L}_{b,c,m}$ *on $C_c(X)$ for a locally finite graph (b, c) over (X, m).*

(ii) *$\mathcal{A}$ satisfies a maximum principle, i.e., if $f \in C_c(X)$ has a non-negative local maximum in $x \in X$, then*

$$\mathcal{A}f(x) \geq 0.$$

Exercise 4.3. *Let X be a countable set, $b : X \times X \to [0, \infty)$, and define $\mathcal{Q}\colon C(X) \to [0, \infty)$ by*

$$\mathcal{Q}(f) = \frac{1}{2} \sum_{x,y \in X} b(x, y)\big(f(x) - f(y)\big)^2.$$

Show the following equivalence:

(i) $\mathcal{Q}(\varphi) < \infty$ *for all $\varphi \in C_c(X)$.*

(ii) $\sum_{y \in X} b(x, y) < \infty$ *for all $x \in X$.*