Lecture 3: Exercises

Exercise 3.1 (Complete graphs). *Let $X = \{1, \dots, N\}$ for some $N \in \mathbb{N}$. The* complete graph *with standard weights (b_K, c_K) with N vertices is given by*

$$b_K(x, y) = 1 \quad \text{for all } x, y \in X \text{ with } x \neq y,$$

and $c_K = 0$. Take your favourite number N with $N \geq 5$.

(a) Draw the graph.

(b) Write down the matrix l_{b_K, c_K} of the Laplacian $L_{b_K, c_K, m}$.

(c) Let $m = 1$. Determine the eigenvalues and a set of orthonormal eigenfunctions of the Laplacian $L_{b_K, c_K, m}$.

(d) Compute the resolvents (and if you wish, also the semigroup).

(e) Solve the Dirichlet problem for $B = \{1, \dots, \lceil N/2 \rceil\}$ and $g = 1_B$, i.e., find $u \in C(X)$ s.t. $L_{b_K, c_K, m} u = 0$ on $X \setminus B$ and $u = g$ on B.

Exercise 3.2 (Bonus: Star graphs). *Let $X = \{0, 1, \dots, N\}$ for some $N \in \mathbb{N}$. The* star graph *with standard weights (b_S, c_S) with $N + 1$ vertices is given by*

$$b_S(0, x) = b_S(x, 0) = 1 \quad \text{for all } x \in X,\ x \neq 0,$$

and

$$b_S(x, y) = 0 \quad \text{for all } x, y \neq 0,$$

with $c_S = 0$. Take your favourite number N with $N \geq 5$.

(a) Draw the graph.

(b) Write down the matrix l_{b_S, c_S} of the Laplacian $L_{b_S, c_S, m}$.

(c) Let $m = 1$. Determine the eigenvalues and a set of orthonormal eigenfunctions of the Laplacian $L_{b_S, c_S, m}$.

(d) Compute the resolvents (and if you wish, also the semigroup).

(e) Solve the Dirichlet problem for $B = \{0, 1, \dots, \lceil N/2 \rceil\}$ and $g = 1_B$.

Exercise 3.3 (Bonus: Fractional Laplacian). *Let X be finite. For a given graph (b, c) over (X, m), let $L = L_{b,0,m}$ be the positive self-adjoint operator on $\ell^2(X, m)$ associated with the graph with semigroup $(e^{-tL})_{t \geq 0}$.*

For $\sigma \in (0, 1)$, the discrete fractional Laplacian *L^σ on $\ell^2(X, m)$ is defined via the spectral theorem. Then L^σ satisfies*

$$L^\sigma f(x) = \frac{1}{|\Gamma(-\sigma)|} \int_0^\infty \frac{(I - e^{-tL}) f(x)}{t^{1+\sigma}} \, dt.$$

(a) Show that (b, c) is stochastically complete (i.e., $e^{-tL} 1 = 1$ for all $t \geq 0$) if and only if $c = 0$.

(b) Assume that $c = 0$. Show that there is a graph (b_σ, c_σ) associated with L^σ.

(c) Assume that $c = 0$. Show that the graph (b_σ, c_σ) is complete.

Solution to 3.3 (a): Remark 1.22 in the lecture notes. Solution to 3.3 (b), (c): see Section 5 in U. Das, M. Keller and Y. Pinchover, On Landis' conjecture for positive Schrödinger operators on graphs. Int. Math. Res. Not. 2025, No. 12, Article ID rnaf151, 20 p. (2025; Zbl 08098263)