Extremal processes and number partitioning

Processus des valeurs extrêmes et partitions de nombres

Anton Bovier [a] Irina Kurkova [b]

[a] *Weierstrass–Institut für Angewandte Analysis und Stochastik*
Mohrenstrasse 39, D-10117 Berlin, Germany, and
Institut für Mathematik
Technische Universität Berlin
Strasse des 17. Juni 136, 10623 Berlin, Germany
[b] *Laboratoire de Probabilités et Modèles Aléatoires*
Université Paris 6
place Jussieu, B.C. 188
75252 Paris, Cedex 5, France

Abstract

We provide a general criterion of convergence of a point process to a Poisson point process. We apply this criterion to the study of the number partitioning problem to distribute N i.i.d. random variables into k groups (of size N/k) in such a way that the sums of the variables in each group are as similar as possible. In the case $k = 2$ it has been shown that the properly rescaled differences of the two sums converge to a Poisson point process on $\mathbb{R}_+$ as $N \to \infty$, as if they were independent random variables. Applying our criterion, we generalize this result to the case $k > 2$: we show that the normalised vector of $k-1$ differences between the k sums converges to a Poisson point process on $\mathbb{R}_+^{k-1}$.

Résumé Nous fournissons un critère général de convergence d'un processus ponctuel vers un processus de Poisson. Nous appliquons ce critère à l'étude d'un problème de répartition des entiers : répartir N variables aléatoires i.i.d. en k groupes (de taille N/k) tels que les sommes des éléments dans les groupes soient aussi similaires que possible. Pour le cas $k = 2$, il fût demontré que les différences des deux sommes, proprement normalisées, convergent vers un processus ponctuel de Poisson sur $\mathbb{R}_+$ pour $N \to \infty$. En utilisant notre critère, on généralise ce résultat pour le cas $k > 2$: on démontre que les vecteurs normalisés des $k-1$ différences consécutives entre les k sommes convergent vers un processus ponctuel de Poisson sur $\mathbb{R}_+^{k-1}$.

Email addresses: `bovier@wias-berlin.de` (Anton Bovier), `kourkova@ccr.jussieu.fr` (Irina Kurkova).

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1. Version française abrégée

La plupart des critères connus qui assurent la convergence d'un processus des valeurs extrêmes vers un processus de Poisson au-delà du cas des variables aléatoires i.i.d. demande soit l'indépendance, la stationarité et des conditions de mélange (voir [8]), ou l'échangeabilité et une version forte d'indépendance asymptotique des marginales de dimension finie [6,5]. Ici on démontre un résultat général sans supposer aucune de ces hypothèses. L'utilité de ce théorème se manifestera sur une application à un problème classique de répartition de nombres.

Soient $\boldsymbol{V}_{i,M} = (V_{i,M}^1, \dots, V_{i,M}^p) \in \mathbb{R}_+^p$, $i = 1, \dots, M$ des vecteurs aléatoires.

Notation. Nous notons par $\sum_{\alpha(l)}$ la somme sur toutes les suites *ordonnées* d'indices *distincts* $\{i_1, \dots, i_l\} \subset \{1, \dots, M\}$. Nous écrivons également $\sum_{\alpha(r_1),\dots,\alpha(r_R)}(\cdot)$ si la somme porte sur toutes les suites *ordonnées* de sous-ensembles ordonnés et disjoints $\alpha(r_1) = (i_1, \dots, i_{r_1})$, $\alpha(r_2) = (i_{r_1+1}, \dots, i_{r_2}), \dots, \alpha(r_R) = (i_{r_1+\dots+r_{R-1}+1}, \dots, i_{r_1+\dots+r_R})$ de $\{1, \dots, M\}$.

Theorem 1.1 *Soit pour tout $l = 1, 2, \dots$ fini et tout ensemble de constantes $c_j^\beta > 0$, $j = 1, \dots, l$, $\beta = 1, \dots, p$*

$$\sum_{\alpha(l)=(i_1,\dots,i_l)} \mathbb{P}\Big(V_{i_j,M}^\beta < c_j^\beta \;\forall j = 1, \dots, l, \beta = 1, \dots, p\Big) \to \prod_{\substack{j=1,\dots,l\\ \beta=1,\dots,p}} c_j^\beta, \quad M \to \infty. \tag{1}$$

Alors, le processus ponctuel sur $\mathbb{R}_+^p$

$$\Pi_M^p = \sum_{i=1}^M \delta_{(V_{i,M}^1,\dots,V_{i,M}^p)} \tag{2}$$

converge faiblement, quand $M \uparrow \infty$, vers le procesus de Poisson sur $\mathbb{R}_+^p$ de mesure d'intensité donnée par la mesure de Lebesgue.

On applique ce resultat à un problème classique de répartition de nombres.

Soient $X_1, \dots, X_N$ des variables aléatoires i.i.d. distribuées selon la loi uniforme sur $[0,1]$. On cherche à répartir l'ensemble $\{1, \dots, N\}$ en k sous-ensembles disjoints $I_1, \dots, I_k$, tels que les sommes $\sum_{n \in I_i} X_n$, $i = 1, \dots, k$, soient les plus proches possibles les unes des autres.

Dans [9,10,1], ce problème s'inscrit dans le contexte de systèmes desordonnés de la mécanique statistique en identifiant les répartitions avec des variables de spins de Potts $\sigma : \{1, \dots, N\} \to \{1, \dots, k\}^N$ modulo les permutations de valeurs $1, \dots, k$. On introduit alors le vecteur $\boldsymbol{Y}(\sigma) = \{Y^\beta(\sigma)\}_{\beta=1}^{k-1}$ de composantes

$$Y^\beta(\sigma) = \sum_{n=1}^N X_n(1\!\!1_{\{\sigma_n=\beta\}} - 1\!\!1_{\{\sigma_n=\beta+1\}}), \quad \beta = 1, \dots, k-1, \tag{3}$$

qui mesure le degré d'uniformité de la répartition σ.

Nous considérons ici le problème restreint qui suppose que la cardinalité de chaque répartition est N/k. On note l'espace de configurations correspondantes par Σ_N. On a

$$|\Sigma_N| = \binom{N}{N/k}\binom{N(1-1/k)}{N/k} \cdots \binom{2N/k}{N/k}(k!)^{-1} \sim k^N (2\pi N)^{\frac{1-k}{2}} k^{\frac{k}{2}} (k!)^{-1} \equiv S(k,N). \tag{4}$$

Theorem 1.2 *Soient*

$$V^\beta(\sigma) = k^{\frac{N}{k-1}} (2\pi N)^{-1} k^{\frac{2k-1}{2k-2}} (k!)^{\frac{-1}{k-1}} 2\sqrt{6}|Y^\beta(\sigma)|, \qquad \beta = 1, \dots, k-1.$$

Alors, le processus ponctuel sur $\mathbb{R}_+^{k-1}$

$$\sum_{\sigma \in \Sigma_N} \delta_{(V^1(\sigma),\dots,V^{k-1}(\sigma))} \tag{5}$$

converge faiblement vers le processus de Poisson sur $\mathbb{R}_+^{k-1}$ de mesure d'intensité donnée par la mesure de Lebesgue.

2. A general extreme value theorem

Most of standard criteria for convergence of point processes to Poisson processes that go beyond the i.i.d. case either assume independence, stationarity, and some mixing conditions (see [8]), or exchangeability and a very strong form of asymptotic independence of the finite dimensional marginals [6,5]. Here we prove a simple general criterion of convergence making no one of these assumptions. We will demonstrate its use in the study of the classical number partitioning problem in the next section.

Consider series of M random vectors $\boldsymbol{V}_{i,M} = (V_{i,M}^1, \ldots, V_{i,M}^p) \in \mathbb{R}_+^p$, $i = 1, \ldots, M$.
Notation. We write $\sum_{\alpha(l)}$ when the sum is taken over all possible *ordered* sequences of *different* indices $\{i_1, \ldots, i_l\} \subset \{1, \ldots, M\}$. We also write $\sum_{\alpha(r_1),\ldots,\alpha(r_R)}(\cdot)$ when the sum is taken over all possible *ordered* sequences of disjoint ordered subsets $\alpha(r_1) = (i_1, \ldots, i_{r_1})$, $\alpha(r_2) = (i_{r_1+1}, \ldots, i_{r_2}), \ldots, \alpha(r_R) = (i_{r_1+\cdots+r_{R-1}+1}, \ldots, i_{r_1+\cdots+r_R})$ of $\{1, \ldots, M\}$.
Theorem 2.1 *Assume that for all finite $l = 1, 2, \ldots$ and all set of constants $c_j^\beta > 0$, $j = 1, \ldots, l$, $\beta = 1, \ldots, p$ we have*

$$\sum_{\alpha(l)=(i_1,\ldots,i_l)} \mathbb{P}\Big(V_{i_j,M}^\beta < c_j^\beta \ \forall j = 1, \ldots, l, \beta = 1, \ldots, p\Big) \to \prod_{\substack{j=1,\ldots,l \\ \beta=1,\ldots,p}} c_j^\beta, \quad M \to \infty. \tag{6}$$

Then the point process

$$\Pi_M^p = \sum_{i=1}^M \delta_{(V_{i,M}^1,\ldots,V_{i,M}^p)} \tag{7}$$

on $\mathbb{R}_+^p$ converges weakly as $M \to \infty$ to the Poisson point process $\mathcal{P}^p$ on $\mathbb{R}_+^p$ with the intensity measure which is the Lebesgue measure
Sketch of the proof. Denote by $\Pi_M^p(A)$ the number of points of the process Π_M^p in a subset $A \subset \mathbb{R}_+^p$. The proof of this theorem follows from Kallenberg theorem [7] on the week convergence of a point process Π_M^p to the Poisson process Π^p. Applying his theorem in our situation weak convergence holds whenever
(i) For all cubes $A = \prod_{\beta=1}^p [a^\beta, b^\beta)$ of volume $|A| = \prod_{\beta=1}^p (b^\beta - a^\beta)$:

$$\mathbb{E}\Pi_M^p(A) \to |A|, \quad M \to \infty. \tag{8}$$

(ii) For all finite union $A = \bigcup_{l=1}^L \prod_{\beta=1}^p [a_l^\beta, b_l^\beta)$ of disjoint cubes of volume $|A| = \sum_{l=1}^L \prod_{\beta=1}^p (b_l^\beta - a_l^\beta)$:

$$\mathbb{P}(\Pi_M^p(A) = 0) \to \epsilon^{-|A|}, \quad M \to \infty. \tag{9}$$

Our main tool of checking (i) and (ii) is the inclusion-exclusion principle: for any $l = 1, 2, \ldots$, any events $O_1, \ldots, O_l$ and any $n \leq [l/2]$:

$$\sum_{k=0}^{2n} \sum_{\substack{\mathcal{A}_k=(i_1,\ldots,i_k) \\ \subset\{1,\ldots,l\} \\ i_1<i_2<\cdots<i_k}} (-1)^k \mathbb{P}\Big(\bigcap_{j=1}^k \bar{O}_{i_j}\Big) \geq \mathbb{P}\Big(\bigcap_{i=1,\ldots,l} O_i\Big) \geq \sum_{k=0}^{2n+1} \sum_{\substack{\mathcal{A}_k=(i_1,\ldots,i_k) \\ \subset\{1,\ldots,l\} \\ i_1<i_2<\cdots<i_k}} (-1)^k \mathbb{P}\Big(\bigcap_{j=1}^k \bar{O}_{i_j}\Big) \tag{10}$$

where $\bar{O}_{i_j}$ are complementary events to O_{i_j}. Combining (10) and the assumption (6), we can prove that, for any subsets $A_1, \ldots, A_l \in \mathbb{R}_+^p$ that can be represented as unions of disjoint cubes, under the hypothesis of Theorem 2.1 we have: for any $m_1, \ldots, m_l$

$$\sum_{\alpha(m_1),\alpha(m_2),\ldots,\alpha(m_l)} \mathbb{P}(\boldsymbol{V}_{i,M} \in A_j \ \forall i \in \alpha(m_r), \forall r = 1,\ldots,l) \to \prod_{r=1}^{l} |A_r|^{m_r}. \tag{11}$$

In particular, with $l = 1$, $m_1 = 1$, this proves condition (i). To verify (ii), let us construct a cube $B = \prod_{\beta=1}^{p}[0, \max_{l=1,\ldots,L} b_l^{\beta})$ of volume $|B|$, then $A \subset B$. For any $R > 0$ we have the decomposition:

$$\mathbb{P}(\Pi_M(A) = 0) = I_1(R, M) + I_2(R, M)$$
$$\equiv \sum_{r=0}^{R} \frac{1}{r!} \sum_{\alpha(r)} \mathbb{P}(\boldsymbol{V}_{i,M} \in B \setminus A \,\forall i \in \alpha(r),\ \boldsymbol{V}_{i,M} \notin B \,\forall i \notin \alpha(r)) + \mathbb{P}(\Pi_M(A) = 0, \Pi_M(B) > R).$$

Applying the inclusion-exclusion principle (10) to $M - r$ events $\{\boldsymbol{V}_i \notin B\}$ for $i \notin \alpha(r)$, we may bound $I_1(R, M)$ for all $n \leq [(M - r)/2]$ by

$$\sum_{r=0}^{R} \frac{1}{r!} \sum_{k=0}^{2n} \frac{(-1)^k}{k!} \sum_{\alpha(r),\alpha(k)} \mathbb{P}(\boldsymbol{V}_{i,M} \in B \setminus A \ \forall i \in \alpha(r),\ \boldsymbol{V}_{i,M} \in B \ \forall i \in \alpha(k)) \geq I_1(R, M)$$
$$\geq \sum_{r=0}^{R} \frac{1}{r!} \sum_{k=0}^{2n+1} \frac{(-1)^k}{k!} \sum_{\alpha(r)\alpha(k)} \mathbb{P}(\boldsymbol{V}_{i,M} \in B \setminus A \ \forall i \in \alpha(r),\ \boldsymbol{V}_{i,M} \in B \ \forall i \in \alpha(k)).$$

Then for any fixed $n \geq 1$, the statement (11) applied to the subsets A/B and B imply:

$$\sum_{r=0}^{R} \frac{|B \setminus A|^r}{r!} \sum_{k=0}^{2n} \frac{(-1)^k |B|^k}{k!} \geq \lim_{M\to\infty} I_1(R, M) \geq \sum_{r=0}^{R} \frac{|B \setminus A|^r}{r!} \sum_{k=0}^{2n+1} \frac{(-1)^k |B|^k}{k!}.$$

It follows that $\lim_{M\to\infty} I_1(R, M) = e^{-|B|} \sum_{r=0}^{R} \frac{|B\setminus A|^r}{r!}$, which is as close to $e^{|B\setminus A| - |B|}$ as desired if R is chosen large enough. Finally, by (11)

$$\lim_{M\to\infty} I_2(R, M) \leq \lim_{M\to\infty} \mathbb{P}(\Pi_M^1(B) > R) = \lim_{M\to\infty} \frac{1}{R!} \sum_{\alpha(R)} \mathbb{P}(\boldsymbol{V}_{i,M} \in B \ \forall i \in \alpha(R)) = \frac{|B|^R}{R!}$$

which is as close to zero as desired if R is chosen large enough. Hence, $\lim_{M\to\infty} \mathbb{P}(\Pi_M^1(A)) = e^{-|A|}$.

3. Application to the number partitioning problem.

The number partitioning problem is a classical problem from combinatorial optimization. We consider N i.i.d. random numbers $X_1, \ldots, X_N$ taken with the uniform distribution on $[0, 1]$. One seeks to partition the set $\{1, \ldots, N\}$ into k disjoint subsets $I_1, \ldots, I_k$, such that k sums $\sum_{n\in I_i} X_n$, $i = 1, \ldots, k$, are as similar to each other as possible. This problem can be cast into the language of mean field spin systems [9,10,1] by realizing that the set of partitions is equivalent to the set of Potts spin variables $\sigma : \{1, \ldots, N\} \to \{1, \ldots, k\}^N$ modulo permutation of the values $\{1, \ldots, k\}$. The vector $\boldsymbol{Y}(\sigma) = \{Y^{\beta}(\sigma)\}_{\beta=1}^{k-1}$ with components

$$Y^{\beta}(\sigma) = \sum_{n=1}^{N} X_n (\mathbb{1}_{\{\sigma_n = \beta\}} - \mathbb{1}_{\{\sigma_n = \beta+1\}}), \quad \beta = 1, \ldots, k - 1, \tag{12}$$

measures the differences of the sums of the X_n in each subset, i.e. the quality of the partition.

We will consider the restricted problem where the cardinality of each I_i has to be equal to N/k. This corresponds to the state space of configurations σ of N spins, such that the number of spins taking each value equals N/k, i.e. $\#\{n : \sigma_n = \beta\} = N/k$ for all $\beta = 1, \ldots, k$. Finally, we must take equivalence classes of these configurations: each class includes $k!$ configurations obtained by a permutation of the values of spins $1, \ldots, k$. We denote by Σ_N the state space of these equivalence classes. Then

$$\|\Sigma_N\| = \binom{N}{N/k}\binom{N(1-1/k)}{N/k}\cdots\binom{2N/k}{N/k}(k!)^{-1} \sim k^N (2\pi N)^{\frac{1-k}{2}} k^{\frac{k}{2}} (k!)^{-1} \equiv S(k,N). \tag{13}$$

Theorem 3.1 *Let*

$$V^\beta(\sigma) = k^{\frac{N}{k-1}} (2\pi N)^{-1} k^{\frac{2k-1}{2k-2}} (k!)^{\frac{-1}{k-1}} 2\sqrt{6} |Y^\beta(\sigma)|, \qquad \beta = 1, \ldots, k-1. \tag{14}$$

Then the point process on $\mathbb{R}_+^{k-1}$

$$\sum_{\sigma \in \Sigma_N} \delta_{(V^1(\sigma), \ldots, V^{k-1}(\sigma))} \tag{15}$$

converges weakly to the Poisson point process on $\mathbb{R}_+^{k-1}$ *with the intensity measure which is the Lebesgue measure.*

Remark 1 The analogous result in the particular case $k = 2$ was proven by Borgs, Chayes and Pittel [3](see also [2]). The difficulty, one is confronted with when proving Theorem 2.1, is that in the situation at hand, standard criteria of convergence to the Poisson process do not apply. We certainly do not have independence or stationarity, nor do we have exchangeability. Worse, also the asymptotic factorization of marginals does not hold uniformly in the form required e.g. in [5].

The proof of Theorem 3.1 involves two steps. In the previous section we announced an abstract theorem that gives criteria for the convergence of an extremal process to a Poisson process. In the next lemma we show that these are satisfied in the problem at hand.

Lemma 3.2 *Let*

$$S(k,N) = k^N (2\pi N)^{\frac{1-k}{2}} k^{\frac{k}{2}} (k!)^{-1} \tag{16}$$

be borrowed from (13). We denote by $\sum_{\sigma^1,\ldots,\sigma^l \in \Sigma^N}(\cdot)$ *the sum over all possible ordered sequences of different elements of* Σ^N. *Then for any* $l = 1, 2, \ldots,$ *any constants* $c_j^\beta > 0$, $j = 1, \ldots, l$, $\beta = 1, \ldots, k-1$ *we have:*

$$\sum_{\sigma^1,\ldots,\sigma^l \in \Sigma_N} \mathbb{P}\Big(\forall \beta = 1, \ldots, k-1, \forall j = 1, \ldots, l \frac{|Y^\beta(\sigma^j)|}{\sqrt{2(N/k)\mathrm{var}\ X}} < \frac{c_j^\beta}{S(k,N)^{\frac{1}{k-1}}}\Big) \to \prod_{\substack{j=1,\ldots,l \\ \beta=1,\ldots,k-1}} (2(2\pi)^{-1/2} c_j^\beta). \tag{17}$$

Unfortunately, for a few pairs of configurations σ, σ', there appear very strong correlations between the vectors $\mathbf{Y}(\sigma)$ and $\mathbf{Y}(\sigma')$ that have to be dealt with. This makes the proof of Lemma 3.2 very tedious. Such a problem did already appear in a milder form in the work [3] for $k = 2$, but in the general case the associated linear algebra problems get seriously more difficult. In the unconstrained problem (that is partitioning $X_1, \ldots, X_N$ into k groups not necessarily of size N/k) this is even worse and has so far prevented us to solve that case.

We will give here only an informal sketch of the proof of Lemma 3.2. The random variables $\frac{Y^\beta(\sigma^j)}{\sqrt{2(N/k)\mathrm{var}\, X}}$ are the sums of independent identically distributed random variables with mean zero and the covariance matrix $B(\sigma^1, \ldots, \sigma^l)$ with the elements

$$b_{i,s}^{\beta,\gamma} = \frac{\mathrm{cov}\,(Y^\beta(\sigma^i), Y^\gamma(\sigma^s))}{2(N/k)\mathrm{var}\, X} = \frac{\sum_{n=1}^N (1\!\!1_{\{\sigma_n^i = \beta\}} - 1\!\!1_{\{\sigma_n^i = \beta+1\}})(1\!\!1_{\{\sigma_n^s = \gamma\}} - 1\!\!1_{\{\sigma_n^s = \gamma+1\}})}{2(N/k)}. \tag{18}$$

In particular:

$$b_{i,i,}^{\beta,\beta} = 1, \;\; b_{i,i}^{\beta,\beta+1} = -1/2, \;\; b_{i,i}^{\beta,\gamma} = 0 \text{ for } \gamma \neq \beta, \beta+1, \; \forall\, i = 1, \ldots, k-1. \tag{19}$$

Moreover, the property that $b_{i,j}^{\beta,\gamma} = o(1)$ as $N \to \infty$ for all $i \neq j$, β, γ, holds for a number $R(N,l)$ of sets $\sigma^1, \ldots, \sigma^l \in \Sigma_N^{\otimes l}$ which is $R(N,l) = \|\Sigma_N\|^l(1+o(1)) = S(k,N)^l(1+o(1))$ with $o(1)$ exponentially small as $N \to \infty$. For all such sets $\sigma^1, \ldots, \sigma^l$, by the Central Limit Theorem, the random variables $\frac{Y^\beta(\sigma^j)}{\sqrt{2(N/k)\text{var}\, X}}$ should behave asymptotically as centered Gaussian random variables with covariances $b_{i,j}^{\beta,\gamma} = 1_{\{i=j,\beta=\gamma\}} + (-1/2)1_{\{i=j,\gamma=\beta+1\}} + o(1)$. The determinant of this covariance matrix is $1+o(1)$. Hence, the probability $\mathbb{P}(\cdot)$ defined in (17) that these Gaussians belong to the exponentially small segments $[-c_j^\beta S(k,N)^{-1/(k-1)}, c_j^\beta S(k,N)^{-1/(k-1)}]$ is of the order $\prod_{\substack{j=1,\ldots,l\\ \beta=1,\ldots,k-1}} (2(2\pi)^{-1/2} c_j^\beta S(k,N)^{-1/(k-1)})$. Multiplying this probability by the number of terms $R(N,l)$ we get the result claimed in (17).

Let us turn to the remaining tiny part of $\Sigma_N^{\otimes l}$ where $\sigma^l, \ldots, \sigma^l$ are such that $b_{i,j}^{\beta,\gamma} \not\to 0$ for some $i \neq j$ as $N \to \infty$. Here two possibilities should be considered differently. The first one is when the covariance matrix $B(\sigma^1, \ldots, \sigma^l)$ of $\frac{Y^\beta(\sigma^j)}{\sqrt{2(N/k)\text{var}\, X}}$ is non-degenerate. Invoking again the Central Limit Theorem, the probability $\mathbb{P}(\cdot)$ in this case is of the order

$$(\text{det} B(\sigma^1, \ldots, \sigma^l))^{-1/2} \prod_{\substack{j=1,\ldots,l\\ \beta=1,\ldots,k-1}} (2(2\pi)^{-1/2} c_j^\beta S(k,N)^{-1/(k-1)}). \tag{20}$$

But from the definition of $b_{i,j}^{\beta,\gamma}$ $(\text{det} B(\sigma^1, \ldots, \sigma^l))^{-1/2}$ may grow at most polynomially. Thus the probability $\mathbb{P}(\cdot)$ is about $S(k,N)^{-l}$ up to a polynomial term while the number of sets $\sigma^1, \ldots, \sigma^l$ in this part is exponentially smaller than $S(k,N)^l$. Hence, the contribution of all such $\sigma^l, \ldots, \sigma^l$ is exponentially small.

The case of $\sigma^1, \ldots, \sigma^l$ with $B(\sigma^1, \ldots, \sigma^l)$ degenerate is more delicate. Although the number of such $\sigma^1, \ldots, \sigma^l$ is exponentially smaller than $S(k,N)^l$, the probability $\mathbb{P}(\cdot)$ is exponentially *bigger* than the number $S(k,N)^{-l}$ since the system of $l(k-1)$ random variables $\{Y^\beta(\sigma^i)\}_{\beta=1,\ldots,k-1}^{i=1,\ldots,l}$ is linearly dependent! This requires precise counting of the number of terms in dependence of the degree of degeneracy, in order to show that these contributions are also negligible. In this analysis the restriction to partitions of equal length is crucial. For the details, see [4].

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