Coarse-Graining Techniques for (random) Kac Models*

Anton Bovier[1] and Christof Külske[2]

[1] Weierstraß-Institut für Angewandte Analysis und Stochastik, Mohrenstrasse 39, 10117 Berlin, Germany, and
Mathematisches Institut, Technische Universität Berlin, Strasse des 17. Juni 136, 10623 Berlin, Germany, bovier@wias-berlin.de
[2] Weierstraß-Institut für Angewandte Analysis und Stochastik, Mohrenstrasse 39, 10117 Berlin, Germany, kuelske@wias-berlin.de

We review our recent results on the low temperature behavior of Kac models. We discuss translation-invariant models and the Kac version of the random field model. For the latter we outline, how various coarse-graining techniques can be used to prove ferromagnetic ordering in dimensions $d \geq 3$, small randomness, and low temperatures, uniformly in the range of the interaction.

1 Introduction

Mean field theory is one of the standard tools of statistical mechanics to get a fast first insight into the behaviour of a complex interacting system. The most prominent example in this context is of course the famous van der Waals theory for the liquid vapour transition. However, mean field models have some undesirable "non-physical" properties, in particular they give rise to non-convex thermodynamic potentials, and as a consequence non-monotonous relations between intensive variables and their conjugate fields. These pathologies can be ad-hoc cured by the Maxwell construction, which simply consists in replacing the non-convex potentials by their convex hulls. To make sense of this procedure as an asymptotic theory for realistic physical models, Kac et al. [17] proposed a model with long, but finite, range interactions (of the form $J_\gamma(r) \equiv \gamma^d J(\gamma r)$, with J a function with bounded support or rapid decay), known as the Kac model. Taking the infinite volume limit for such a model first, and then considering the limit as the range of interactions tends to infinity while appropriately rescaling the interaction strength, one then recovers mean field theory. The most precise and complete form of this asymptotic relation was later proven by Lebowitz and Penrose [23]. They showed that the

* This work was supported on part by the DFG priority program 1033

rate function for the total mean magnetization in the Kac model converges, in the limit of infinite interaction range, to the convex hull of the corresponding rate function in the Curie-Weiss model. Such results were later recovered for more complicated mean field models, such as the Curie-Weiss-Potts model (see e.g. [16] for a survey).

While these results show that mean field theory can provide reasonable free energies, the issue remains whether finer result, and in particular the phase structure of the model on the level of the Gibbs measures is properly represented. Obviously this cannot in general be true, since mean field theory does not see the influence of dimensionality. How is this reflected in the properties of the Gibbs measures of the corresponding Kac models? If we take, for instance the Kac version of the ferromagnetic Ising model, it is clear that for any value of the parameter γ, if $d = 1$, there exists a unique Gibbs state, while the mean field model has two extremal Gibbs states if the inverse temperature, β, is smaller than the critical values 1. The question to what extend a refined analysis of the Gibbs measures of the Kac models allows one to see some trace of the mean field phase transition was addressed in a seminal paper by Cassandro, Orlandi, and Presutti [13]. They showed that on suitably chosen mesoscopic scales the local mean magnetization under the Gibbs measure of the Kac-Ising model is concentrated sharply near the values of the two mean field magnetisations $\pm m^*(\beta)$ (if γ is small). Moreover, typical magnetization profiles are constant near one of the two values over lengths of the order of $\exp(\gamma^{-1}\beta)$.

In higher dimension, one would expect that the convergence of the Gibbs measures to the mean-field limit is even more straightforward since now more than two extremal states can exist already for finite γ. A natural conjecture would be that the critical temperature $\beta_c(\gamma)$ in all such models should converge to that of the mean field model, as $\gamma \downarrow 0$. Such a result was proven in the case of the Ising model in $d \geq 2$ only rather recently in Cassandro, Marra, and Presutti [9] and Bovier and Zahradník [10]. Both proofs rely very heavily on the spin flip symmetry of the model, and therefore do not generalize easily to a wider class of systems.

Studying Kac versions of models with more complex interactions is actually very attractive since one can hope to take advantage of the fact that the model is "close" to a mean field model in some sense. This may allow, for instance, to estimate the local entropy of configurations close to candidate ground states. This has lead us to be interested in Kac versions of certain types of disordered spin systems whose mean field versions can be rigorously analysed. A particularly attractive model in this respect are the Hopfield models. On the level of large deviation results à la Lebowitz-Penrose, this program has been carried out successfully in [2]. Pushing the analysis towards the Gibbs measures, however, proved much harder. In $d = 1$ it was shown in [3] that local profiles again concentrate near the admissible mean field values, and lower bounds on the typical length over which these remain constant were given. Some finer results were later proven in the simpler case of a random field Kac-Ising model by Cassandro, Orlandi and Picco [12].

The analysis of higher dimensional disordered models has proven to be a major challenge due to the surprising lack of adequate techniques. The development of such tools has been a central theme of the research project that is being reported here. Let us mention that parallel to our efforts, the idea of taking advantage of Kac interactions has been very successfully implemented in a different context, namely that of the liquid vapour transition in a particle system in continuous space. Defining a model with a rather particular form of a four body Kac-interaction, Lebowitz, Maazel, and Presutti [22] have given the first proof of such a phase transition in a single type particle system in continuous space.

The basic difficulties encountered in the analysis of the Gibbs measures of Kac models at low temperatures comes from the fact that the basic methods of low temperature expansions are all devised for models with predominantly strong and short range interaction. In such a situation it is possible to devise perturbative expansions around a single "ground state configuration". In Kac models, where the interactions are very weak, but long range, such an expansion must invariably fail (unless $\beta = O(\gamma^{-d})$). As all methods to analyse the phase structure of lattice models rely on such expansion techniques, there one of the first tasks is the development of suitable methods of low temperature expansion in such a situation. This was done in [11] (and also in [22]).

In the present paper we review our results about the Gibbs measures of (mostly) long-range models. The most difficult part of the analysis concerns the proof of the existence of ferromagnetic order in the random field Kac model for a range of temperatures that is uniform in the Kac parameter γ, for dimensions $d \geq 3$. We cannot give full proofs here but we will explain the various coarse-graining procedures that are involved. This is done in a sequence of steps. Our main philosophy is the *reduction of the long-range random field model to an effective discrete short range contour model* on large enough scales. The latter can then be treated by the renormalization group techniques of Bricmont and Kupiainen. This strategy was already successfully applied to prove ferromagnetic ordering in the continuous spin random field model in [18].

To start, in Section 2 we describe how to define contours in long-range models. The contour definition we give is somewhat subtle [11],[15], but it is needed, even in translation-invariant models, for a satisfactory understanding of the low-temperature phases. These contours will be thick on the order of $1/\gamma$. In Section 3 we review the known results about short-range random field models. We first outline of the RG-treatment of the short range random field Ising model in [8], see also [6]. Then we sketch how the continuous spin random field model can be reduced to the discrete case, as a first successful application of the *reduction* strategy.

In Section 4 we give our main result about the random field Kac Ising model and provide some ideas about the key steps of the proof. The coarse-graining to the nearest neighbor model builds on the contour techniques of section 2, but there is some new difficulty that necessitates the introduction of an additional coarse-graining on an even larger scale than the range of the interaction.

2 Translation-invariant long-range models

An important technical tool for the analysis of the phase structure of spin systems are reformulations in terms of contour models and convergent cluster expansions. In the case of Kac models, there are some fundamentally new effects that required a substantial reformulation of the classical theory. In [11] this was developed for a rather broad class of predominantly "ferromagnetic" models with weak long range interactions. The challenge here is to develop convergent expansions for a range of temperatures that is *independent of the range of the interactions*, but depends only on the "total strength" of the dominant part of the attractive interaction.

Specifically, we consider Hamiltonians of the form

$$H(\sigma) = \sum_{\{i,j\}} \Phi^{\gamma}_{i-j}(\sigma_i, \sigma_j) + \sum_i U(\sigma_i) \tag{1}$$

with σ_i taking values in a finite set Q. The interaction kernel Φ^{γ}_{i-j} is assumed to have finite range $1/\gamma$, to be roughly constant on its range and to verify $\sum_k \|\Phi^{\gamma}_k\|_\infty = O(1)$. We don't assume any symmetry under permutation of the spin-values.

In this context [11] developed a contour representation for the partition function that allows to compute relevant physical quantities in terms of convergent expansions for a range of temperatures that is uniform in γ as γ tends to 0. As expanded on in [24], this allows to extend the full power of the Pirogov-Sinai theory to such model, and in particular to give a rigorous analysis of the phase diagram in this range of temperatures.

In the remainder of this section we will explain the main ideas in a simplified context that will be relevant for the application in the random field Kac model.

2.1 $1/\gamma$-contour model representation

The main step of the proof is a reformulation of the long-range model in terms of a suitable contour model. We call it $1/\gamma$-contour model because the resulting contours will be thick on the scale of the range of the interaction. Remember that our main emphasis lies on the fact that we are able to treat models without permutation symmetry under the spins. However, for the sake of a transparent explanation of the basic features that are due to the long-range nature, let restrict ourselves just to the simplest non-trivial long-range model, i.e. let us specialize to the Ising model with spin-flip symmetric translation-invariant interaction. We remark that the formulae to be discussed here have immediate random generalizations to the random field Kac model, to be discussed in Section 4.

So, let us look at the Hamiltonian

$$H(\sigma) = -\sum_{\substack{\{i,j\}\\ i\neq j}} J_\gamma(i-j)\sigma_i\sigma_j \tag{2}$$

The spin variables $\sigma = (\sigma_i)_{i\in\mathbb{Z}^d}$ take values in $\{-1,1\}^{\mathbb{Z}^d}$. We take the simplest possible choice for the couplings being the indicator function of a cube of sidelength $R = \gamma^{-1} \in \mathbb{N}$ and the energy-difference due to the pair interaction of flipping one spin in a sea of plusses is exactly equal to one, that is

$$J_\gamma(i) = \frac{1_{U_R^\bullet}(i)}{|U_R^\bullet|} \qquad \text{where} \qquad U_R^\bullet = \{j \neq 0, |j| \leq R\} \tag{3}$$

The main observation is that, in contrast to a short range model, it is impossible to expand the model around the perfect plus- resp. minus-groundstates. Applying the standard PS-theory for short-range models would only yield ferromagnetic ordering for temperatures that decrease with the range of the interaction tending to infinity. This problem is cured (very roughly) in the following way: 1) Replace the perfect infinite volume plus- or minus-groundstate by sets of perturbed plus- or minus-like configurations ("ensembles") that are characterized by a low density of wrong signs. These ensembles can be treated by high-temperature expansions. 2) Define the contours as the complement of those regions, that is those regions of space where the spins are not plus- resp. minus-like. Show that these contours obey Peierls estimates and their ensembles can be treated by methods known from short-range Pirogov-Sinai theory. While this is the correct main idea, a contour definition in terms of just one density constraint following the above procedure literally is a little too simplistic, and needs some modification.

Contours of a configuration and Peierls bounds

We would like to include a correct definition of a contours and low-density ensembles for the Ising model, although it seems technically slightly involved at first sight. The version of the contour definition coincides essentially with the one given in [11]; however we present the version given in [15] that is also used in [7].

Fix a density threshold $\delta > 0$ and a sign $s = \pm 1$. We call i a (δ, s)-*correct point* of a configuration σ if

$$\#\{j : j - i \in U_R^\bullet, \sigma_j \neq s\} \leq \frac{\delta}{2}\,|U_R^\bullet| \tag{4}$$

Note that this definition makes no reference of the sign at the site i itself; it is only a property of the configuration around it. So, let us define the *cleaned configuration* $\bar{\sigma} = \bar{\sigma}(\sigma)$ corresponding to σ by setting

$$\bar{\sigma}_i(\sigma) = \begin{cases} s & \text{if } i \text{ is } (\delta, s) - \text{correct for } \sigma \\ \sigma_i & \text{else,} \end{cases} \tag{5}$$

We denote by $I_\delta(\sigma)$ the set of δ-incorrect point of the configuration σ. This set would be a natural first guess for the support of the contours to be defined. This is clear because deviation from almost homogeneous spin-configurations should come at a high energetic price. However, in order to arrive at a successful decomposition of the partition function into low-density ensembles and contours a little more care is needed.

First of all, as usual in short-range PS-theory, a *contour* $\Gamma = (\underline{\Gamma}, \sigma_{\underline{\Gamma}})$ is by definition a pair given by a connected subset $\underline{\Gamma} \subset \mathbb{Z}^d$, called the *support*, and a spin configuration $\sigma_{\underline{\Gamma}}$ on this subset. Here we will define supports of contours that are thick of the order of the range of the interaction. To be explicit, partition $\mathbb{Z}^d$ into disjoint cubes of side length $l > 2R$ whose centers lie of the sites of a square sub-lattice of $\mathbb{Z}^d$, and assume that $l = \nu R$, $\nu > 2$. Denote the set of all subsets of $\mathbb{Z}^d$ that can be written as unions of such cubes (with fixed sublattice!) by $C^{(l)}$. Denote by $[A]_R = \{j, |j - i| \leq R, i \in A\}$ the R-neighborhood of a set A.

Now, take a second density threshold value $\tilde{\delta} < \delta$.

Definition 2.1 *Let σ be a spin-configuration. We call the connected components of*

$$I^*(\sigma) := \bigcap_{\substack{M \in C^{(l)},\, M \supset [I_\delta(\sigma)]_R \\ M \supset [I_{\tilde{\delta}}(\sigma_M \bar{\sigma}_{M^c})]_R}} M \tag{6}$$

the support of the contours of the configuration σ. A system of contours $\overline{\Gamma} = \{\Gamma_1, \ldots, \Gamma_n\}$ is admissible if there is a spin-configuration σ such that $\overline{\Gamma}$ is the associated system of contours.

Denote by $\Lambda_s(\sigma)$ the set of (δ, s)-correct points. Then one has that $d(\Lambda_+(\sigma), \Lambda_-(\sigma)) > l$ and $\mathbb{Z}^d = I^*(\sigma) \cup \Lambda_+(\sigma) \cup \Lambda_-(\sigma)$. It is not too difficult to see that the intersection is over non-empty sets M and $I^*(\sigma)$ is in fact the smallest set M having the three properties appearing in the last intersection. This is done explicitly in [15] Lemma 2.2. ff.

If we want to control the finite volume Gibbs measures with plus boundary conditions we have to look at the partition function in finite volume Λ with plus boundary conditions. In the following the summation over all spin-configurations will be split in the following way. 1) Sum over all possible sets $I^*(\sigma)$, and the possible spin configurations on those sets. 2) Conditional on those sets and spin-configurations sum over all spin-configurations that are compatible with them.

This procedure gives the following decomposition into a sum over compatible contours and partition functions of ***restricted low-density*** ensembles.

$$\sum_{\sigma_\Lambda} e^{-\beta H^+_\Lambda(\sigma_\Lambda)} = \sum_{\overline{\Gamma}} \Big(\prod_{\Gamma} \rho(\Gamma)\Big)\, Z^{+,r}_{\Lambda_+}(+_{\Lambda^c}, \sigma_{\underline{\Gamma}})\, Z^{-,r}_{\Lambda_-}(\sigma_{\underline{\Gamma}}) \tag{7}$$

For the partition functions of the restricted ensembles one has the explicit formula

$$\begin{aligned} Z^{s,r}_\Lambda(\sigma_{\Lambda^c}) = \sum_{\sigma_\Lambda} \Bigg(\prod_{i\in[\Lambda]_R} 1_{i \text{ is } (\delta,s) \text{ correct for } \sigma} \Bigg) \\ \times \exp\Bigg(2\beta \sum_{\substack{\{i,j\}\cap\Lambda\neq\emptyset \\ i\neq j}} J_\gamma(i-j) 1_{\sigma_i\neq s} 1_{\sigma_j\neq s} - \beta \sum_{i\in\Lambda} 1_{\sigma_i\neq s} \Bigg) \end{aligned} \tag{8}$$

They depend on boundary conditions imposed by the spin-configurations $\sigma_{\underline{\Gamma}}$ on the contours (and the plus boundary condition outside of Λ). The contour activities $\rho(\Gamma)$ simply collect the terms of the interactions that are only between sites located on the support of the contour. (Here the interaction between contour-sites and the cleaned configuration put on the sites outside of the contour is included, too.)

A main virtue of the contour-definition is that the following Peierls estimate holds.

Lemma 1. *There is a dimension-dependent constant $c = c(\tilde\delta)$ such that*

$$\rho(\Gamma) \le e^{-c(\tilde\delta)\beta|\underline{\Gamma}|} \tag{9}$$

Note the important fact that the volume $|\underline{\Gamma}|$ is always a multiple of l^d and so we have a rather strong suppression. How can we understand that this volume-suppression is always on the scale of the range of the interaction? It is because once there are incorrect points for some density, there are even Const R^d of them, for a slightly lowered density, and each of these contributes an amount of energy that is bounded below by a positive constant. We don't give a full proof here.

Cluster Expansion of restricted "low density" ensembles

The next important step of the preparation of the model is the following: The restricted partition function (8) can be written as a polymer partition function

$$Z_{\Lambda}^{s,r}(\sigma_{\Lambda^c}) = \sum_{(P_1,\dots,P_n)_{cp}} \prod_{i=1}^{n} \tilde{w}_{P_i} \tag{10}$$

with polymers P_i that interact only via volume constraints. The logic to achieve this is as follows: Expand the interaction term between any pair of sites i, j in (8) first. We note that each site with the wrong sign $-s$ is energetically suppressed, due to the second term in the second exponential in (2.9). So, there is sufficient energetic suppression of a configuration as soon as there are not too many of such spins cooperating, so that the pair-inteaction term is not too big. This is shown using the indicator functions in the first brackets. Next, the indicator functions describing the density-constraint are expanded. This is needed in order to produce a gas of non-interacting polymers. Their activities are then shown to obey nice estimates and allow for a cluster expansion.

Because of their geometry these polymers were given the name *galaxies* in [11]. To be a little more precise, the polymers are of the form $P = (t_1, \dots, t_l, N_1, \dots, N_k)$ where t_i are trees connecting the sets N_i. The trees t_i arise from the high-temperature expansion in the restricted ensembles. The sets N_i are connected components of the form $\cup_j B_R(j)$. They arise from the expansion of the non-local density constraints. One has for the polymer activities $\tilde{w}_P$ of galaxies P the estimates

$$\sum_{P:V(P)\ni i} |\tilde{w}_P| e^{\text{const}\,\beta|V(P)|} \le \varepsilon \tag{11}$$

where $V(P) = \bigcup_t V(t) \cup \bigcup_i N_i$.

From (11) follows that one may now perform a cluster expansion of (10) and exponentiate the restricted partition function. We stress again at this point that the same constructions work for non-symmetric models with finite state space.

The resulting representation is then the starting point for the further treatment of such model by short-range Pirogov-Sinai type techniques. This is explained in [24].

3 Random short range models and Coarse-grainings

To achieve our final goal and treat the Kac random field Ising model we will find an effective short-range model that describes the long range model on a sufficiently large scale. For this short-range model ferromagnetic order is well-established [8] and can be carried over to the original model. This philosophy of "reduction to an effective short-range Ising type model" was already devised in [18] to treat the nearest continuous spin model. This will be shortly discussed in Subsection 3.2. However, at first we must turn to a review of the short-range model itself.

3.1 The random field Ising model

So, let us consider the random field Ising model (with symmetric non-degenerate distribution) and nearest neighbor interaction whose Hamiltonian is given by

$$H[\eta](\sigma) = -\sum_{\langle i,j \rangle} \sigma_i \sigma_j - \sum_i \eta_i \sigma_i \tag{12}$$

Here the $(\eta_i)_{i \in \mathbb{Z}^d}$, are i.i.d. symmetrically distributed random variables that satisfy the probabilistic bound

$$\mathbb{P}\left[\eta_i \geq t\right] \leq e^{-\frac{t^2}{2\sigma^2}} \tag{13}$$

where the $\sigma^2 \geq 0$ governing the smallness of the random variables has to be sufficiently small.

We are then interested in the random Gibbs measures

$$\mu[\eta](d\sigma) \propto e^{-\beta H[\eta](\sigma)} d\sigma \tag{14}$$

where $d\sigma$ is the symmetric product measure on the spin configurations σ. (Of course, this is to be understood as a solution to the DLR-equations written down for fixed random field configuration η.)

For this model it was proved in [1] that there is unicity of the Gibbs measure in 2-dimensions, at any fixed temperature, for $\mathbb{P}$-a.e. η. This is in contrast to the case of the model without disorder, which shows that the introduction of arbitrarily weak random perturbations can destroy a phase transition. It shows that randomness can potentially alter the behavior of the system in a fundamental way, and cannot always be treated as a small perturbation. We remark that their (martingale-)method was later applied by [5] to show the non-localization of interfaces in random environments in the framework of certain models for interfaces without overhangs in space dimensions less than $3 = 2 + 1$.

In the three or more dimensional random field Ising model, for small disorder, and small temperature, however, disorder does *not* destroy the ferromagnetic ordering. Here, Bricmont and Kupiainen showed in their famous paper [8] that there exist Gibbs measures $\mu^+[\eta]$ (and $\mu^-[\eta]$), which, for typical magnetic field configuration η, describe small perturbations around a plus-like (respectively a minus-like) infinite-volume ground state. A plus-like ground state looks like a sea of pluses with rare islands of minuses in those regions of space where the realizations of the magnetic fields happen to be mostly oriented to favor the minus spins. We remark that the result of [8] was a nice example where a question that was truly under debate among theoretical physicists could be settled by mathematicians.

Contour model representation of short-range random field model - renormalization group

The method they used to control the Gibbs measures, the so-called 'renormalization group', is a multiscale method that consists in a successive application of a certain coarse-graining/rescaling procedure. This is necessary because there is no simple Peierls-condition for this model (say around the all-plus state.) The individual steps are controlled by expansion methods and probabilistic estimates of the undesirable event that regions of exceptionally large magnetic fields occur. This has to be done for all hierarchies occurring. This method is conceptually beautiful but technically hard to implement. It was later also applied by [6] to show the stability of certain interface models in dimensions $d+1 \geq 4$.

What is the Bricmont-Kupiainen renormalization group in a little more detail? First of all, one has to find a representation of the model as a short-range contour model. Here a contour is again a pair $\Gamma = (\underline{\Gamma}, \sigma_{\underline{\Gamma}})$ of a support and a spin-configuration on the support. Then the RG-transformation of the model is carried out in the contour representation in finite but arbitrarily large volume, for every fixed realization of the disorder η. This RG-transformation roughly consists of two steps: Fix an integer L describing the blocklength. First step: Integration of small contours with diameter smaller than $L/4$. The resulting expression for the partition function contains only a sum over the remaining large contours with new interaction produced by the small-contour sum. Second step: Do a coarse-graining of the remaining large contours. This eliminates the "wiggles" of the remaining large contours so that the resulting model can then be rescaled. The result is a model that lives again on a lattice with the original lattice spacing.

It is the virtue of Bricmont and Kupiainen to have found a representation of the model that is stable under this transformation for all hierarchies, along with a suitable type of bounds. Running the RG an iterative proof shows that a) temperature effectively becomes smaller and b) that randomness becomes smaller and thus more irrelevant on large hierarchies. This translates into the fact that the formation of large contours is very unlikely. This proves ferromagnetic ordering. The same basic structure underlies the work on the stability of SOS-interfaces by the authors.

Now, let us provide some more details about the representation of the contour model that is "BK-renormalizable", that is invariant under the renormalization procedure sketched above. These details will be needed to understand some key features of the proof for the Kac random field model sketched in Section 4. For a spin configuration and a collection of contours $\overline{\Gamma}$ we denote by $\sigma_i(\overline{\Gamma})$ the spin configuration that equals σ_i on $\underline{\Gamma}$ and equals s for $i \in \Lambda_s$. Denote by $Z_\Lambda[\eta]$ the partition function of the model after an arbitrary number of steps of the renormalization group transformation. Then, for each realization of η, and after each finite number of steps one always has the representation

$$
\begin{aligned}
Z_\Lambda[\eta] = &\sum_{\overline{\Gamma}} \Big(\prod_{\Gamma} \rho[\eta](\Gamma) \Big) \\
&\times \exp\Big(\sum_{i \in \Lambda} S_i(\eta)\sigma_i[\overline{\Gamma}] + \sum_{C \in \Lambda_+(\overline{\Gamma})} S_C^+(\eta) + \sum_{C \in \Lambda_-(\overline{\Gamma})} S_C^-(\eta) + \sum_C W_C^{\overline{\Gamma}}(\eta) \Big)
\end{aligned}
\tag{15}
$$

Here $S_i(\eta)$ is a local random field at the site i, which depends on the original random field η only in a stricly local way. It obeys a bound of the type (13) with a new "renormalized (upper bound on the) variance" σ^2. The quantity σ^2 depends on the hierarchy and decreases with application of the RG. $S_C^\pm(\eta)$ are nonlocal random fields, depending on ***connected sets*** C on the lattice of at least 2 points. They are exponentially suppressed in the volume,

$$
|S_C^\pm(\eta)| \leq e^{-\mathrm{const}\,\tilde{\beta}|C|} \tag{16}
$$

with some hierarchy-dependent constant $\tilde{\beta}$ and keep the symmetry of the model, $S_C^+(\eta) = -S_C^-(-\eta)$. When we will try to apply the renormalization group strategy to the long range model this geometric structure of the C 's being ***connected sets*** will become important and cause some additional complexities.

The $\rho[\eta](\Gamma)$ are contour activities, depending on the realization of the random fields. ***Essentially*** they obey deterministic Peierls bounds $0 \leq \rho[\eta](\Gamma) \leq e^{-\mathrm{const}\,\beta|\underline{\Gamma}|}$, β being a "renormalized inverse temperature". However the Peierls bound just stated holds only for contours in regions of space where the underlying randomness was not "too big". A failure of the simple Peierls bound may happen in the so called large field region. Without being explicit about this let us mention that it is one important part of the proof to show that this large field region is very exceptional and becomes less and less important under RG. Finally, $W_C^{\overline{\Gamma}}[\eta]$ are nonlocal interactions between the contours; they are nonzero only for connected sets C that intersect the support of the contours and obey deterministic upper bounds of the same type as the nonlocal random fields. They are just correction terms whose creation under RG can't be avoided.

The hard technical work of the RG-analysis then consists in showing that this form is invariant, with $\beta \uparrow \infty$, $\tilde{\beta} \uparrow \infty$, $\sigma^2 \downarrow 0$ and the bad region dying out under successive application of the RG.

3.2 The continuous spin random field model

In the context of disordered systems the continuous spin version corresponding to the random field model is an important model to study. In the physics literature is no less popular than the Ising model itself, so it is interesting to see whether ferromagnetic ordering can be proved for this model, too. Now the spin variables m_i take values in $\mathbb{R}$ and the formal Hamiltonian for a spin-configuration $m \in \mathbb{R}^{\mathbb{Z}^d}$ in the infinite volume is given by

$$H[\eta](m) = \frac{q}{2} \sum_{\langle i,j \rangle} (m_i - m_j)^2 + \sum_i V(m_i) - \sum_i \eta_i m_i \tag{17}$$

where the first summation extends over all pairs of nearest neighbors $\langle i, j \rangle$. The finite volume Gibbs measures are then obviously given by taking the exponential of the negative finite volume restriction of (17) as the non-normalized Lebesgue-density. The potential V has a symmetric double-well structure. We will stick to the most popular choice which is a polynomial of fourth order. Let us choose a scaling where the potential has unity curvature in the minima $\pm m^*$ that is

$$V(m_i) = \frac{\left(m_i^2 - (m^*)^2\right)^2}{8 m^{*2}} \tag{18}$$

and investigate the Gibbs measures for $q \geq 0$ sufficiently small and $q\,(m^*)^2$ sufficiently large. The latter quantity gives the order of magnitude of the minimal energetic contribution to the Hamiltonian (17) caused by neighboring spins in different wells. Thus it corresponds to a Peierls constant. Moreover we impose a fixed uniform bound on $|\eta_i|$, independent of σ^2. This is for technical reasons.

Since we are dealing with continuous degrees of freedom a contour definition and a direct application of the renormalization group in the framework of Bricmont and Kupiainen is not immediate, and if possible would entail a huge amount of technical work. Moreover one might be afraid that the additional continuous degrees of freedom could lead to a possible loss of order.

In this context we show that there is in fact a 'ferromagnetic' phase transition, in dimensions $d \geq 3$, for sufficiently small σ^2 (meaning small disorder), sufficiently large $q(m^*)^2$, and not too big $q(m^*)^{\frac{2}{3}}$ (controlling the 'anharmonicity' of the minima, as it can be seen from the proof). We prove the following: The [random] Gibbs-probability (w.r.t. to the finite volume-measure with plus-boundary conditions) of finding the spin left to the positive potential well is very small, uniformly in the volume, on a set of realizations of η of a size [w.r.t $\mathbb{P}$] of at least $1 - e^{-\frac{\text{const}}{\sigma^2}}$. The precise statement is found in Theorem 1 p.1272 of [18]. For more information and explanation we refer to the introduction of [18]. Let us however mention the following: The particular form of the potential as a fourth order polynomial is of no importance, as well as the requirement of uniform boundedness on the random fields and the restriction to nearest neighbor couplings in the Hamiltonian (instead of finite-range interactions) could be given up.

Reduction to short-range contour model - Coarse-graining in spin-space

The novelty of the proof is the use of a stochastic mapping of the continuous spins to their sign-field (independently over the sites). We choose it such that the probability that a continuous spin m_x is mapped to its sign is given by $\frac{1}{2}(1+\tanh(am^*|m_i|))$. (Here a is a parameter close to one that needs to be tuned in a useful way.) The image measure of a particular sign-configuration then gives the approximate weights of finding continuous spins in the neighborhood of the potential wells indexed by these signs. Using a suitable combination between high temperature and low temperature expansions it is shown that the resulting model has the form of an Ising model with exponentially decaying interactions. (These expansions are related to those used by [Za00] in the translation-invariant context where however, due to the lack of positivity, no probabilistic interpretation can be given.) This can be seen as a 'single-site-coarse-graining'-method. Next, having constructed the Ising-system, it can be cast into a contour model representation for which the renormalization group of [BrKu88] can be used. We remark that a lot of technical work is needed to implement this idea of the reduction to a discrete model. Still, while doing so, there is still a great deal of work saved which was already done on the level of the discrete RG.

For the readers interested in the theory of generalized Gibbs measures we also remark that this mapping is really compatible with the infinite volume limit in the sense that the infinite system under consideration is mapped to a proper infinite volume *Gibbs* measure of an Ising model (see Theorem 2 of [18], p.1273). So, this stochastic map also provides an interesting example of a 'coarse-graining without pathologies'. This means that the coarse-graining produces no 'artificial' non-local dependencies in the conditional expectations of the resulting measure.

Let us just mention at this place that, in contrast to that, disordered systems frequently provide a source of various non-Gibbsian measures when we look at them jointly on the product space of disorder variables and spin variables. This is known as the so-called Morita-approach in theoretical physics. For more rigorous research on this we refer to [19, 20, 21] and the references therein.

4 The random field Kac Ising model

Let us finally turn to the discussion of our main result. Consider the model with Hamiltonian

$$H_\gamma[\eta](\sigma) = -\sum_{\substack{\{i,j\}\\ i\neq j}} J_\gamma(i-j)\sigma_i\sigma_j - \sum_i \eta_i\sigma_i$$

Then we prove the following.

Theorem 4.1 *Assume that $d \geq 3$. Assume that η_i are symmetrically distributed i.i.d. and $|\eta_i| \leq \delta_{RF}$. Then there is $\gamma_0 > 0$ and $\beta_0 < \infty$, such that for all $\gamma \leq \gamma_0$ and $\beta \geq \beta_0$ we have*

$$\mathbb{E}\Big(\mu^+_{\beta,\gamma}[\eta](\sigma_i = 1)\Big) > \frac{1}{2}.$$

We will sketch two steps of the proof. In the first step, described in 4.1 we will obtain a contour-representation that is quite analogous to the one described for the translation-invariant Kac-model in Subsection 2.1. Unfortunately the result is not yet good enough to yield a formulation of the model allows for the RG-treatment. We will then briefly indicated what is the problem and the cure in Subsection 4.1.

4.1 $1/\gamma$-contour model representation

Let us do the decomposition into contours and low-density ensembles without any reference to the configuration of the the random field η.

The first main point is the following decomposition that generalizes (1) to the random situation

$$\begin{aligned} Z^{\text{Kac}}_{\gamma,\Lambda}[\eta] &\equiv \sum_{\sigma_\Lambda} e^{-\beta H^+_\Lambda[\eta](\sigma_\Lambda)} \\ &= \sum_{\overline{\Gamma}} \Big(\prod_{\Gamma} \rho(\Gamma)\Big) \exp\left(\beta \sum_{i\in\Lambda} \eta_i \sigma_i[\overline{\Gamma}]\right) Z^{+,r}_{\Lambda_+}[\eta_{\Lambda_+}](+_{\Lambda^c}, \sigma_{\underline{\Gamma}})\; Z^{-,r}_{\Lambda_-}[\eta_{\Lambda_-}](\sigma_{\underline{\Gamma}}) \end{aligned} \tag{19}$$

Here the contour-activities are identical to those in the non-random case described in (8) but the restricted partition functions acquire the modified form

$$\begin{aligned} Z^{s,r}_{\Lambda}[\eta_\Lambda](\sigma_{\Lambda^c}) &= \sum_{\sigma_\Lambda} \left(\prod_{i \in [\Lambda]_R} 1_{i \text{ is } (\delta,s) \text{ correct for } \sigma} \right) \\ &\times \exp\left(2\beta \sum_{\substack{\{i,j\}\cap\Lambda\neq\emptyset \\ i\neq j}} J_\gamma(i-j) 1_{\sigma_i \neq s} 1_{\sigma_j\neq s} - \beta \sum_{i\in\Lambda} (1+2s\eta_i) 1_{\sigma_i\neq s} \right) \end{aligned} \tag{20}$$

They can be written as partition functions for a polymer gas in the very same way as for the translation-invariant model. Of course, now the polymer activities will depend on the random fields. Under the condition of uniform boundedness of the random fields with δ_{RF} sufficiently small there is however still sufficient suppression of the polymer-activities. Performing the cluster expansion for these partition functions we obtain the representation

$$Z^{\mathrm{Kac}}_{\gamma,\Lambda}[\eta] = \sum_{\overline{\Gamma}} \Big(\prod_{\Gamma} \rho(\Gamma) \Big)$$
$$\times \exp\left(\beta \sum_{i\in\Lambda} \eta_i \sigma_i[\overline{\Gamma}] + \sum_{\substack{\mathcal{C} \\ V(\mathcal{C})\subset\Lambda_+(\overline{\Gamma})}} w^+_{\mathcal{C}}(\eta) + \sum_{\substack{\mathcal{C} \\ V(\mathcal{C})\subset\Lambda_-(\overline{\Gamma})}} w^-_{\mathcal{C}}(\eta) + \sum_{\mathcal{C}\mathrm{icp}\,\overline{\Gamma}} w^{\overline{\Gamma}}_{\mathcal{C}}(\eta) \right) \tag{21}$$

The symbols $\mathcal{C}$ denote clusters of the type of polymers defined below (10), and $V(\mathcal{C})$ is just the union of the volumes $V(P)$ over the polymers P the cluster is made of. From a uniform version of (11) than follows that the corresponding fields obey the uniform exponential estimate

$$\sum_{\mathcal{C}:V(\mathcal{C})\ni i} \sup_{\eta} |w^{+,-,\overline{\Gamma}}_{\mathcal{C}}(\eta)| e^{\mathrm{const}\,\beta V(\mathcal{C})} \le \mathrm{Const} \tag{22}$$

At first glance this looks close to the form (15) that is invariant under the Bricmont-Kupiainen renormalization group, but it is not! This is due to the fact that the clusters $\mathcal{C}$ are made of polymers whose geometric structure is quite different from the connected sets occurring for the short-range random field model. We will see that problems are occurring if we perform one additional RG-step of the Bricmont-Kupiainen renormalization. The problems are due to the fact that in the RG-invariant formulation of the model *connected* sets appear as indices of the non-local contributions of the field while we are dealing here with possibly spread out geometric objects. Remember that the clusters are made of polymers which contain trees arising from the expansion of the long-range interactions. The simplest and "most spread-out" polymer occurring is just a string composed of bonds whose length are of the order of the range of the interaction R. The most straightforward idea to create corresponding connected sets would therefor be to perform one additional blocking-step with blocks of side-length of the order R. However, as we will see below this won't provide us with the desired bounds that allow for further application of the RG for a range of inverse temperatures β that is uniform in the range of interaction. To explain the problem and its cure a little more in detail let us be more general and see what happens under a blocking with block-length LR, L being a constant to be determined.

4.2 *LR*-blocking

Let us denote by x a block on the RL-lattice and put for the *local part* of the effective random field associated to this block the trial definition

$$S_x(\eta) := \sum_{i\in x} \beta\eta_i + \sum_{\substack{\mathcal{C}:V(\mathcal{C})\subset x \text{ or} \\ d(\mathcal{C})<LR/4}} \frac{w^+_{\mathcal{C}}(\eta) - w^-_{\mathcal{C}}(\eta)}{2n(\mathcal{C})} \tag{23}$$

where $n(\mathcal{C})$ is the number of blocks on the RL-lattice that are intersected by the volume of the cluster $V(\mathcal{C})$. This is a straightforward and natural construction in the RG-treatment.

The remaining sums over clusters of galaxies have to be blocked by infection. They give rise to a non-local small field term S_K that is indexed by connected subsets K of adjacent blocks on the RL-lattice.

$$S_K^{\pm}(\eta) := \sum_{\substack{\mathcal{C}:\mathcal{C}\mapsto K \text{ and} \\ d(\mathcal{C})\geq (RL)/4}} w_{\mathcal{C}}^{\pm}(\eta) \tag{24}$$

where we sum over those clusters $\mathcal{C}$ for which K is the minimal set of adjacent RL-blocks that contains their volume $V(\mathcal{C})$.

The local term poses no difficulties, it is the non-local problem that is difficult. Indeed, to treat the first, we note that the variation of the function $S_x(\eta)$ w.r.t. change of a single random field is at most $\beta\delta_{RF} + e^{-\text{const}\,\beta}$. Since $S_x(\eta)$ depends only on η_i in a set of sites that is of the order $(RL)^d$ one gets by a well-known martingale estimate the desired Gaussian upper bound of the form (13)

$$\mathbb{P}\left[S_x \geq t\right] \leq e^{-\text{const}\,\frac{t^2}{(\beta\delta_{RF}+e^{-\text{const}\,\beta})^2(RL)^d}} \tag{25}$$

for all $t \geq 0$.

Let us now try to get an exponential bound of the type (16) for the non-local part (24), where $|C|$ has to be replaced by the rescaled volume, which is just the number of blocks $n = |K|/(RL)^d$. The best we can do is to estimate

$$|S_K^{\pm}| \leq \sum_{i\in K} \sum_{\substack{\mathcal{C}:V(\mathcal{C})\ni i \text{ and } \mathcal{C}\mapsto K \\ \text{and } d(\mathcal{C})\geq (RL)/4}} |w_{\mathcal{C}}^{\pm}(\eta)| \leq n(RL)^d e^{-\text{const}\,\beta L n} \tag{26}$$

Here the factor L in the exponential stems from the fact that the "worst" clusters that are contributing to the sum have a minimal volume of the order L. This is already seen by looking at polymers made of trees with bonds with width of the order R. The prefactor $n(RL)^d$ counts the number of "anchoring points" i of such clusters. This bound is not unnatural because the cluster-sum corresponds to a sort of free energy which should be of the order of the volume, measured on the original scale. This shows that this bound can not easily be improved. Now, the problem is that the prefactor causes this bound to be non-uniform in R.

However, for any fixed L, the exponential dominates the prefactor as soon as we demand that $\beta \geq \text{Const}\log R$. In fact, in this range of temperatures the full preparation of the model to accommodate for the short-range RG is easy and we get ferromagnetic ordering without too much difficulty.

However, the reason why we made L explicit in the above formulas is that we want to make it R-dependent according to $L(R) = \text{Const} \log R$.

Then, for $\beta \geq \beta_0$ sufficiently large, where β_0 is *uniform in R*, we get the desired estimate

$$|S_K^{\pm}| \leq e^{-\mathbf{const}\,\beta L(R) n} \tag{27}$$

The present argument only gives an outline why we are forced to deal with an additional length-scale. It remains to show that a model (15) with desired bounds can be rigorously derived, incorporating the influence of contours whose diameter is smaller than $RL/4$. There is a fair amount of technical work involved. The details and additional arguments can be found in [7].

References

1. M. Aizenman and J. Wehr, "Rounding effects on quenched randomness on first-order phase transitions", Commun. Math. Phys. **130**, 489 (1990).
2. A. Bovier, V. Gayrard, and P. Picco, "Large deviation principles for the Hopfield model and the Kac-Hopfield model", Prob. Theor. Rel. Fields **101**, 511-546 (1995).
3. A. Bovier, V. Gayrard, and P. Picco, "Distribution of overlap profiles in the one-dimensional Kac-Hopfield model", Comm. Math. Phys. **186**, 323–379 (1997).
4. A. Bovier and C. Külske, "A rigorous renormalization group method for interfaces in random media", Rev. Math. Phys. **6**, 413–496 (1994)
5. A. Bovier and C. Külske, "There are no nice interfaces in $(2+1)$-dimensional SOS models in random media", J. Stat. Phys. **83** (1996), 751-759 (1996).
6. A. Bovier and Ch. Külske, A rigorous renormalization group method for interfaces in random media, Rev. Math. Phys. **6**, 413-496 (1994).
7. A. Bovier, C. Külske, *Phase Transition in the three dimensional random field Kac model*, in preparation
8. J. Bricmont, and A. Kupiainen, "Phase transition in the 3d random field Ising model", Commun. Math. Phys. **116**, 539-572 (1988).
9. M. Cassandro, R. Marra, and E. Presutti, "Corrections to the critical temperature in 2d Ising systems with Kac potentials", J. Stat. Phys. **78**, 1131-1138 (1995).
10. A. Bovier and M. Zahradník, "The low-temperature phase of Kac-Ising models". J. Stat. Phys. **87**, 311–332 (1997).
11. A. Bovier and M. Zahradník, "Cluster expansions and Pirogov–Sinai theory for long range spin systems", Markov Proc. Rel. Fields **8**, 443-478 (2002).
12. M. Cassandro, E. Orlandi, P. Picco, "Typical configurations for one-dimensional random field Kac model", Ann. Probab. **27**, 1414–1467 (1999)

13. M. Cassandro, E. Orlandi, and E. Presutti, "Interfaces and typical Gibbs configurations for one-dimensional Kac potentials", Prob. Theor. Rel. Fields **96**, 57-96 (1993).
14. A. De Masi, E. Orlandi, E. Presutti, and L. Triolo, "Glauber evolution with Kac potentials, I. Mesoscopic and macroscopic limits, interface dynamics", Nonlinearity **7**, 633-696 (1994); Glauber evolution with Kac potentials. II. Fluctuations. Nonlinearity **9**, 27–51 (1996); Glauber evolution with Kac potentials. III. Spinodal decomposition. Nonlinearity **9**, 53–114 (1996).
15. S. Friedli S., C.-E. Pfister, "Non-Analyticity and the van der Waals limit", preprint (2003), to appear in J. Stat. Phys.
16. H. Kesten and R. Shonmann, "Behaviour in large dimensions of the Potts and Heisenberg model", Rev. Math. Phys. **1**: 147-182 (1990).
17. M. Kac, G. Uhlenbeck, and P.C. Hemmer, "On the van der Waals theory of vapour-liquid equilibrium. I. Discussion of a one-dimensional model" J. Math. Phys. **4**, 216-228 (1963); "II. Discussion of the distribution functions" J. Math. Phys. **4**, 229-247 (1963); "III. Discussion of the critical region", J. Math. Phys. **5**, 60-74 (1964).
18. C. Külske, "The continuous spin random field model: Ferromagnetic ordering in $d \geq 3$", Rev.Math.Phys., **11**, 1269-1314 (1999)
19. C. Külske, "Weakly Gibbsian representations for joint measures of quenched lattice spin models" Probab. Theory Related Fields 119 (2001)
20. C. Külske, "Analogues of non-Gibbsianness in joint measures of disordered mean field models" J. Stat. Phys. **112**, 1079–1108 (2003)
21. C. Külske, A. Le Ny, F. Redig, "Relative entropy and variational properties of generalized Gibbsian measures", Eurandom preprint 2002-035, accepted for publication in Annals of Probability
22. J. L. Lebowitz, E. Presutti, "Liquid-vapor phase transitions for systems with finite-range interactions", J. Stat. Phys. **94**, 955–1025 (1999)
23. J. Lebowitz and O. Penrose, "Rigorous treatment of the Van der Waals Maxwell theory of the liquid-vapour transition", J. Math. Phys. **7**, 98-113 (1966)
24. M. Zahradník, "Cluster expansions of small contours in abstract Pirogov-Sinai models", Markov Process. Related Fields **8**, 383–441 (2002)