MEAN FIELD SPIN GLASSES AND NEURAL NETWORKS *

Anton Bovier†

7th February 2005

Introduction and models Rarely has a paper with a title as innocent as *"A solvable model of a spin glass"* had such a tremendous impact both in physics and in mathematics as the seminal paper of 1972 by Sherrington and Kirkpatrick, that introduced what is now known as the *Sherrington-Kirkpatrick mean field spin glass model.* For solvable as it might have appeared to the authors, it was soon found that the heuristic solution, based on the so-called replica method, was physically unacceptable. The reason was a tacit assumption, now known as *replica symmetry* that proved unfounded. Several years later, Giorgio Parisi provided an ingenious way out through his *continuous replica symmetry breaking scheme*,

* *Keywords:* mean field models, spin glasses, Sherrington-Kirkpatrick model, Hopfield model, Generalized Random Energy model, disordered systems, neural networks, Gaussian processes, replica method, Gibbs measures

†Weierstrass-Institute for Applied Analysis and Stochastics, Mohrenstrasse 39, 10117 Berlin, Germany, and Mathematisches Institut, Technische Universität Berlin, Strasse des 17. Juni 135, 10623 Berlin, Germany

that presented a solution that through its complexity and intrinsic beauty both stunned and fascinated the community. Unravelling the mysteries involved in this solution has presented a challenge and driving force for the last three decades of mathematical statistical mechanics, while the use of the method in theoretical physics opened the path to solving a wide variety of problems not only in the theory of disordered magnets, but also in neural networks and combinatorial optimization. In this article I will focus on the mathematical results obtained in the study of this and a number of related models.

Mean field models have played an important rôle in statistical mechanics by providing simple, solvable models in which some of the complex phenomena such as phase transitions could be studied and understood. Let me mention the Curie-Weiss model of a ferromagnet. It describes N spin variables σ_i taking values ± 1 in interaction. The simplifying assumption compared to more realistic models, such as the Ising model is to forget about the spatial structure of the model and to allow all spins to interact with all others with equal strength. This yields to a Hamiltonian function of the form

$$H_N(\sigma) = -\frac{J}{N} \sum_{i,j=1}^{N} \sigma_i \sigma_j + h \sum_{i=1}^{N} \sigma_i \tag{0.1}$$

where J is a coupling constant and h a magnetic field. This from of the interaction implies that the Hamiltonian is in fact just a function of the empirical magnetization $m_N(\sigma) = N^{-1} \sum_{i=1} \sigma_i$, and this fact allows to use methods from the theory of Large Deviations to analyze rather easily the corresponding Gibbs measures

$$\mu_{\beta,N}(\sigma) \equiv \frac{e^{-\beta H_N(\sigma)}}{Z_{\beta,N}} \tag{0.2}$$

The Sherrington-Kirkpatrick model was a straightforward attempt to introduce a mean field version of models with *randomly* interacting spins. The interest in such models arose from the discovery of certain alloys of ferromagnets and conductors (like AuFe and CuMn) that had been found to exhibit very unusual magnetic properties. Ruderman et al. had proposed that in these models the magnetic ions with magnetic moments S_i and S_j located at the points x_i and x_j would interact via an exchange interaction of the form $\frac{\cos(k_f(x_i-x_j))}{|x_i-x_j|^2} S_i \cdot S_j$. Since the positions of the magnetic ions in the alloy are random, the signs of their interaction would be *oscillatory* . Ph. Anderson proposed a simplified model in the spirit of the Ising model where spins taking values ± 1 located on a regular lattice would interact via nearest neighbor couplings J_{ij} modelled as i.i.d. random variables uniformly distributed on an interval $[-J, J]$. In the spirit of the Curie-Weiss model, Sherrington and Kirkpatrick then proposed the mean field model where any two spins would interact via i.i.d. Gaussian random variables J_{ij} of mean zero and variance one. The SK Hamiltonian is thus given by

$$H_N^{SK}(\sigma) \equiv -\frac{J}{\sqrt{N}} \sum_{1 \leq i < j \leq N} J_{ij} \sigma_i \sigma_j + h \sum_{i=1}^{N} \sigma_i \tag{0.3}$$

where the normalization is chosen to ensure that the variance of H_N is an extensive quantity. If the two Hamiltonians superficially look similar, the main feature allowing to solve the Curie-Weiss model is absent in the SK model: there is no way to write the Hamiltonian as a function of one (or several) macroscopic variables such as the magnetization. This implies that all methods known to solve the Curie-Weiss model fail here. The approach used systematically in the physics literature to overcome this difficulty is to try to compute the *mean free energy* $f_{\beta,N} \equiv -\frac{1}{\beta N} \mathbb{E} \ln Z_{\beta,N}$ using the formal identity $\ln x = \lim_{q \downarrow 0} q{-}1(x^q - 1)$.

For $q \in N$, one easily sees that (we put $h = 0$)

$$\mathbb{E}Z_{\beta,N}^q = \sum_{\sigma^1,\dots,\sigma^q} \exp\left(\frac{\beta^2 J^2}{2} N \sum_{a,b=1}^{q} \sum_{i<j} \sigma_i^a \sigma_i^b \sigma_j^a \sigma_j^b\right)$$

This latter expression looks already more like the partition function of an ordinary mean fields model, and the computation with standard methods seemed feasible. However, passage to the limit $q \downarrow 0$ remains a highly risky enterprise, and it took the genius of Parisi to develop an approach that provided at least a physically meaningful and convincing answer. The replica method being dealt with in another article, I will not explain this approach any further, although we will explain the nature of the result in the light of recent rigorous work later on.

Site disordered models. The difficulties encountered with the random bond interactions led readily to proposals of mean field models that were closer to the Curie-Weiss model from the point of view that they allowed the Hamiltonian to be written as a function of macroscopic variables. The most important of these models was introduced by Figotin and Pastur. Here the disorder was introduced as a M-dimensional vector ξ_i for each site i. The components of this vector are usually taken as i..i.d. random variables ξ_i^μ taking values ± 1 with equal probability. One can then introduce M-dimensional vectors macroscopic variable that generalize the magnetisation with components

$$m_N^\mu(\sigma) \equiv N^{-1} \sum_{i=1}^{N} \xi_i^\mu \sigma_i$$

The Hamiltonian can then be written as

$$\begin{aligned} H_N(\sigma) &= -N \sum_{\mu=1}^{M} \left(m_N^\mu(\sigma)\right)^2 \\ &= -\frac{1}{N} \sum_{i,j=1}^{N} \sigma_i \sigma_j \sum_{\mu=1}^{M} \xi_i^\mu \xi_j^\mu \end{aligned}$$

These models were indeed found to be solvable with tools similar to those used in the Curie-Weiss case; however they proved disappointing in that the solution did not show the features characteristic expected in a spin glass. In fact it turns out the these models behave very much like a mean field ferromagnet instead that they display not just two equilibrium states at low temperatures, but $2M$ of them, concentrated on spin configurations σ for which $m_N(\sigma)$ takes values close to one of the values $\pm m^*(\beta)e_\mu$, where e_μ is the μ- unit vector in $\mathbb{R}^M$ and $m^*(\beta)$ solves the equation $m = \tanh(\beta m)$ known from the Curie-Weiss model. This model might have been forgotten, had it not been rediscovered in 1982 by J. Hopfield in the context of *neural networks.* Hopfield realised that if σ_i are interpreted as the *activation states* ("firing" and "not firing") of neurons in the brain, the form of the interaction in this model is exactly the one proposed earlier by Hebb for synaptic interaction between neurons having "learned" the M "patterns" ξ^μ in the past. He went on to interpret $H_N(\sigma)$ as the Lyapounov function of the retrieval algorithm by which the brain would recognize the learned pattern. Naturally, the fact the the configurations ξ^μ are minima of H_N then implies the functioning of the algorithm. The important observation of Hopfield was that, based on numerical experiments, the algorithm failed when M became too large. In fact he observed a breakdown of the memory if $M \geq 0.14N$. That meant that the interesting asymptotics in this model required to consider M as an increasing function of N. This regime was not covered by large deviation type results and an intensive program to investigate this model was initiated. Again, the replica method could be employed and yielded a very rich structure of the model, including an explanation of the findings of Hopfield. These models also turned out to be an important starting point for the rigorous analysis.

Gaussian processes and Derrida's models. While the models we have seen so far were motivated from the point of view of randomly interacting spins, Derrida had the consequential idea to view the Hamiltonian of such a model simply as a random process indexed by the set of all spin configuration. In the case of the SK model, this process was moreover a Gaussian process and thus characterized entirely by its mean and variance. For $h = 0$ we see that

$$\mathbb{E}H_N^{SK}(\sigma)H_N^{SK}(\sigma') = \frac{N}{2}\left(r_N(\sigma,\sigma')\right)^2 - \frac{1}{2}r_N(\sigma,\sigma')$$

where $r_N(\sigma,\sigma') \equiv N^{-1}\sigma_i\sigma_i'$ is usually called the *overlap*. This opened the view to a much larger class of models. In particular, the simplest model from this perspective corresponds to taking $H_N(\sigma)$ as a process of i.i.d. random variables. Derrida called this the *Random Energy Model* (REM). He also noted that it could be seen as the limit if a sequence of so-called p-spin SK models corresponding to the covariance of the Hamiltonian being $N(r_N(\sigma,\sigma'))^p$. On the other hand, Derrida observed that another class of models could be defined that were easier to analyse while exhibiting much of the complex properties of the SK model. These are obtained by choosing the covariance not as a function of the overlap (resp. the Hamming distance), but of a ultra-metric distance related to $d_N(\sigma,\sigma') \equiv N^{-1}(\inf\{i : \sigma_i \neq \sigma_i'\} - 1)$. These models, called *Generalised Random Energy Models* (GREM) were analysed by Derrida and Gardner in the 80ies and are now the only models where the full predictions of the Paris theory can be rigorously justified. I will return to this in some detail later.

Further models and applications. Where is a wealth of problems that can be interpreted in terms of disordered mean field models, and that may be analysed using methods developed here. I mention only some of the most notable ones that have received more attentions lately: The *perceptron*, a feed-forward neural network was analysed first by E. Gardner using the replica method. Very recently, M. Shcherbina and B. Tirozzi gave a rigorous justification of this result. The *p-satisfiability problem* is an important problem in computer science that also can be analysed with the replica method. Rigorous results are still very limited. The *number partitioning problem* can be formulated as a random energy model. Also the most famous problem in combinatorial optimization can be solved heuristically with the replica method. Further emerging fields are *turbo-codes*.

Formulation of the problem. Given a model, that is a Hamiltonian function defined as a random process, the ultimate goal is to describe the asymptotic properties of the corresponding Gibbs measure, ideally identifying a (random or deterministic) limiting measure, as a function of the temperature, β^{-1} and other parameters, like e.g. the magnetic field h.

The first steps in this direction concern global properties:

- Does the ground state energy density,

$$\lim_{N\uparrow\infty} \max_{\sigma\in\mathcal{S}_N} H_N(\sigma)$$

  converge (in what sense?) and what is the limit?

- What is the limit of the free energy

$$f_{\beta,N} \equiv \frac{-1}{\beta N} \ln Z_{\beta,N}$$

It has been noted in the mid 90ies that such quantities are usually self-averaging in the sense that e.g.

$$\lim_{N\uparrow\infty}(f_{\beta,N}-\mathbb{E}f_{\beta,N})=0,\ \text{a.s}$$

due to the *concentration of measure* phenomenon. However, until very recently the existence of the limits was considered an open problem in most of the models described above. In 2002, F. Guerra and F.L. Toninelli discovered that a clever use of comparison inequalities for convex functions of Gaussian processes allows to prove a priori the existence of limits at least in the case of models based on Gaussian processes (SK, GREM). The main task is the computation of the values of the limit.

If the free energy is known as a function of sufficiently many parameters, one can frequently compute a number of correlation functions that characterize the limiting measure as well. What one should compute is somewhat model dependent.

Geometry of Gibbs measures and multi-overlap distributions. The problem of describing in a satisfactory way the asymptotic geometric properties of random Gibbs measures on $\{-1,1\}$ is rendered difficult as the symmetries of the problem make the use of local topologies seem unattractive. A reasonable way of solving this problem is as follows. Let D_N be a distance on $\mathcal{S}_N$ normalized so that $\max_{\sigma,\tau\in\mathcal{S}_N}D_N(\sigma,\tau)=1$. Then consider the mass distribution around any fixed point σ,

$$m_\sigma(x)\equiv\mu_{\beta,N}(D_N(\sigma,\sigma')\leq x)$$

and construct the biased empirical average

$$\mathcal{K}_{\beta,N}\equiv\sum_{\sigma\in\mathcal{S}_N}\mu_{\beta,N}(\sigma)\delta_{m_\sigma(\cdot)}$$

The set of distributions of these random measures is compact (with respect to the weak topology) and thus we can expect to construct limits. The law of $\mathcal{K}_{\beta,N}$ is fully determined by the family averaged distributions of the distances between n independent copies of σ drawn from the Gibbs measures,

$$\mathbb{E}\mu_{\beta,N}^{\otimes n}(D_N(\sigma^1,\sigma^2),\dots,D_N(\sigma^{n-1},\sigma^n))$$

In the SK models, one chooses $D_N(\sigma,\tau)=1-\frac{1}{N}\sum_i\sigma_i\tau_i$, so that these quantities can be expressed as distributions of the *overlaps* $\frac{1}{N}\sum_i\sigma_i\tau_i$, between n "replicas" spin variables. In the GREM models it is natural to chose as distance the lexicographic distance used in the construction of the models. In this case the limits of $\mathcal{K}_{\beta,N}$ can be constructed explicitly and it was shown that they can be expressed in terms of the size biased empirical family size distribution of a certain continuous state branching process via a model dependent time change. Since this will play a key rôle not only in the GREMs, we will go into some detail to elucidate this structure.

Neveu's process and random genealogies. The random structure of the limiting Gibbs measures of the GREM models (and presumably also the SK models, even though this is not proven) can be traced to a continuous state branching process introduced by J. Neveu, and an induced associated random genealogy on the unit interval. Let Z_t be a time homogeneous continuous time Markov process with state space $\mathbb{R}_+$ characterized by the Laplace transform of its transition kernel

$$\mathbb{E}(e^{-\lambda Z_t}|Z_0=a)=e^{-a\lambda^{e^{-t}}}$$

Based on this process, construct a two parameter process $Z(t,a)$ with the property that for any $a,b>0$; the processes $Z(\cdot,a)$ and $Z(\cdot,a+b)-Z(\cdot,a)$ are independent have the same laws as Z_t with initial conditions a, resp. b. It follows that $Z(t,\dot{)}$ is a stable subordinator with exponent e^{-t}. Now let $\theta_t(a) \equiv \frac{Z(t,a)}{Z(t,1)}$. as a function on $[0,1]$, θ_t is a random probability distribution function (of pure point type). Any such family μ_t of distributions define in a natural way a *genealogical structure* on $[0,1]$. Define the *ancestor* of $\alpha \in [0,1]$ at time $t < 1$ to be $a_t(\alpha) \equiv \theta_t(\theta_1^{-1}(\alpha))$, where θ^{-1} is the right-continuous inverse of the non-decreasing function θ.

We say that for $\alpha, \alpha' \in [0,1]$, $q(\alpha, \alpha') = t$ if and only if $t = \sup\left(s : a_s(\alpha) = a_s(\alpha')\right)$. It is easy to see that $1 - q$ defines an ultra-metric distance. We can associate with this the distribution size of the offspring of an ancestor at time t, $m_\alpha(t) = |\alpha' : q(\alpha, \alpha') \leq t|$, and its size biased empirical distribution

$$\mathcal{K} \equiv \int_0^1 d\alpha \delta_{m_\alpha(\cdot)}$$

In the GREM models it can be shown that the quantity $\mathcal{K}_{\beta,N}$ converges (weakly in law) to the corresponding $\mathcal{K}$ obtained from a time change of the family of measures θ_t, namely

$$\theta_t^{m_\beta} \equiv \theta_{\ln m(t) - \ln m(0)}$$

where m is a non-decreasing function that can be computed explicitly. Namely, if $\mathbb{E} X_\sigma X_\tau = A(d_N(\sigma, \tau)$, and $\bar{a}$ denotes the right-derivative of the concave hull of A, then

$$m(x) = \min\left(\beta^{-1}\sqrt{2\ln 2}/\sqrt{\bar{a}(x)}, 1\right)$$

As we will explain shortly, similar results are expected in the SK models.

Interpolation methods and Guerra's integral representation. A very important tool for the analysis of Gaussian models in particular have been interpolation methods that allow to compare functions of processes with different variance. While these methods go back to early work on Gaussian processes (Slepian, Kahane), they have been employed with remarkable success in the present context. Mostly they consist of introducing an interpolating Hamiltonian $H^t(\sigma) \equiv \sqrt{t}H(\sigma) + \sqrt{1-t}K(\sigma)$ where K is a reference process that has certain desired properties. Given any function F of the process (e.g. the free energy of the model), one then represents

$$F(H) = F(K) + \int_0^1 dt \frac{d}{dt} F(H^t)$$

Often the derivative on the right hand side can be controlled rather well, e.g. because of some obvious positivity properties.

Example 1. (Guerra and Toninelli) Choose

$$\begin{aligned} K(\sigma) &= \frac{1}{\sqrt{M}} \sum_{i<j=1}^{M} J'_{ij}\sigma_i\sigma_j \\ &+ \frac{1}{\sqrt{N-M}} \sum_{i<j=M+1}^{N} J'_{ij}\sigma_i\sigma_j \end{aligned}$$

and consider the free energy $F(H^t) = f^t_{\beta,N}$. Then, first $F(H^0_N) = F(H_M) + F(H_{N-M})$. On the other hand,

$$\begin{aligned} \frac{d}{dt}F(H^t_N) = & - \frac{1}{2N}\mu^t_{\beta,N}\Big(\sum_{i<j=1}^{N} \frac{\sigma_i\sigma_j J_{ij}}{tN} \\ & - \sum_{i<j=1}^{M} \frac{J'_{ij}\sigma_i\sigma_j}{\sqrt{(1-t)M}} + \sum_{i<j=M+1}^{N} \frac{J'_{ij}\sigma_i\sigma_j}{\sqrt{(1-t)(N-M)}} \Big) \end{aligned}$$

A key tool to be used at this stage is the so-called Gaussian integration by parts formula, $\mathbb{E}gf(g) = \mathbb{E}f'(g)$. Applied here it gives

$$\begin{aligned} \frac{d}{dt}F(H^t_N) &= \frac{\beta}{4N^2}\mu^{t,\otimes 2}_{\beta,N}\Big((\sum_{i=1}^{N} \sigma_i\sigma'_i)^2 \\ &- \frac{N}{M}(\sum_{i=1}^{M} \sigma_i\sigma'_i)^2 - \frac{N}{N-M}(\sum_{i=M+1}^{N} \sigma_i\sigma'_i)^2 \Big) \geq 0 \end{aligned}$$

This proves super-additivity of $N\mathbb{E}f_{\beta,N}$,

$$N\mathbb{E}f_{\beta,N} \geq M\mathbb{E}f_{\beta,M} + (N-M)\mathbb{E}f_{\beta,N-M}$$

which in turn implies convergence of $\mathbb{E}f_{\beta,N}$ to a limit $\mathbb{E}f_\beta$. Moreover, standard concentration of measure estimates show then that $f_{\beta,N}$ also converges almost surely.

Example 2. (Guerra, Aizenman-Sims-Starr) A more complicated application of the interpolation method allows to relate the free energy to Parisi's solution. This was first found by F. Guerra, but a different, and in some sense more intuitive formulation was given later by Aizenman, Sims, and Starr. It is based on the following construction. We consider a centered Gaussian process $H_N(\sigma)$ on $\mathcal{S}_N$ with covariance given by $Ng(R_N(\sigma,\sigma'))$ for some even convex function $g : [-1,1] \to [0,1]$. Let us take $F(H_N) = \ln \mathbb{E}_\sigma e^{\beta H_N(\sigma)}$ (the a priori expectation $\mathbb{E}_\sigma$ need not be symmetric, but may incorporate a magnetic field). Before using comparison, we now want to go to a larger space. For this introduce some set $\mathcal{A}$ equipped with some positive definite quadratic form q, normalized such that $q_{\alpha,\alpha} = 1$, and $|q_{\alpha,\alpha'}| \leq 1$, $\forall_{\alpha,\alpha' \in \mathcal{A}}$. Let $\mathbb{P}_\alpha$ denote some probability measure on $\mathcal{A}$. Now introduce a centered Gaussian process κ_α on $\mathcal{A}$, independent of H_N, whose covariance is given by $\mathbb{E}\kappa_\alpha\kappa_{\alpha'} = r(q_{\alpha,\alpha'}) \equiv q_{\alpha,\alpha'}g'(q_{\alpha,\alpha'}) - g(q_{\alpha,\alpha'})$. Define

$$G(H_N + \sqrt{N}\kappa) = \ln\left(\mathbb{E}_\sigma\mathbb{E}_\alpha e^{-\beta(H_N(\sigma)+\sqrt{N}\kappa_\alpha)}\right)$$

Obviously, $G(H_N,\kappa) = F(H_N) + \widetilde{F}(\kappa)$, where $\widetilde{F}(\kappa) = \ln\left(\mathbb{E}_\alpha e^{-\beta\sqrt{N}\kappa_\alpha}\right)$. The amazing idea is now to compare the process $(H_N + \kappa)$ with another process $\eta_{\sigma,\alpha}$ whose covariance is a linear function of $R_N(\sigma)$ (this is in some sense a Slepian's process), and that otherwise is smaller than the covariance of $(H_N + \kappa)$, to wit

$$\mathbb{E}\eta_{\alpha,\sigma}\eta_{\alpha',\sigma'} = R_N(\sigma,\sigma')g'(q_{\alpha,\alpha'})$$

By these choices of covariances one has that for $x \in [-1,1]$, $y \in [0,1]$, since g is even and convex,

$$g(x) + yg'(y) - g(y) \geq xg'(y)$$

It is an immediate consequence of Kahane's theorem, respectively the same interpolation argument given above, that

$$\mathbb{E}G(H_N + \kappa) \leq \mathbb{E}G(\eta),$$

which translates into

$$\mathbb{E}F(H_N) \leq \mathbb{E}G(\eta) - \mathbb{E}\widetilde{F}(\kappa)$$

It is clear that we can optimize this bound by choosing $\mathcal{A}$, q, and P_α. Of course, the difficulty would be to find such a minimum. A first simplification of this optimization problem is to consider instead of the deterministic structure of $\mathbb{P}$ and q random probability measures on the space of probability measures and quadratic forms on $\mathcal{A}$, to average over the preceding equation w.r.t. their laws, and then take the infimum over all such random structure. This gives a (still incalculable) bound that Aizenman et al. have shown to be asymptotically sharp, i.e. they showed that

$$\lim_{N\uparrow\infty} \mathbb{E}F(H_N) = \lim_{N\uparrow\infty} \inf_{\mathcal{A},\mu} \mathbb{E}_\mu(\mathbb{E}G(\eta) - \mathbb{E}\widetilde{F}(\kappa))$$

where μ is short for all probability measures on the space of $(\mathbb{P}_\alpha, q_{\alpha,\alpha'})$ on $\mathcal{A}$ (called 'random overlap structures' in Aizenman et al.). Guerra's bound consists in restricting the infimum to a class of where the bound is calculable 'explicitly'. Maybe unsurprisingly, this is exactly the class of asymptotic models that have already arisen in the GREM's. In fact, we set $\mathcal{A} = [0,1]$, $\mathcal{M} \equiv \{m : [0,1] \to [0,1]$, non-decreasing$\}$, let q be the random genealogical distance associated to the family of measures θ_t^m, and let $\mathbb{P}_\alpha$ be the probability measure on $\mathcal{A}$ whose distribution function is $\theta_1^m(\alpha)$. Then Guerra's bound states that

$$\lim_{N\uparrow\infty} \mathbb{E}F(H_N) \leq \lim_{N\uparrow\infty} \inf_{m\in\mathcal{M}} \mathbb{E}G(\eta) - \mathbb{E}\widetilde{F}(\kappa)$$

where the expectations relate to all random quantities involved. By self-averaging, the same result holds almost surely. The right-hand side of this equation is known as (a particular formulation of) the famous Parisi solution. In fact if we define the function $f(q,y)$ as the solution of the non-linear partial differential equation

$$\partial_q f + \frac{1}{2}\left(\partial_y^2 f + m(q)(\partial_y f)^2\right) = 0$$

with final conditions

$$f(1,q) = \ln\cosh\beta y$$

These equations can be solved by elementary means in the case when m is a step function. It turns out that for given m,

$$\mathbb{E}G(\eta) - \mathbb{E}\widetilde{F}(\kappa)) = f(0,h,m,\beta) - \frac{\beta^2}{2}\int_0^1 qm(q)dr(q)$$

where $h = \beta^{-1}\cosh^{-1}(\mathbb{E}_\sigma\sigma_1)$. This solution was originally obtained using the replica method. The preceding construction gives, at the least, a clear mathematical meaning to the objects involved. In particular, the notion of 'ultra-metric zero-dimensional matrices', appears now to be equivalent to ultra-metric structures on the unit interval.

In a recent paper, M. Talagrand has announced that converse inequality is also true in the preceding equation, confirming that Parisi's solution yields the correct free energy in a large class of models of the SK type..

Ghirlanda-Guerra relations. The appearance of a universal probabilistic structure in the asymptotics of these models may appear surprising. A partial explanation can be found in a set of remarkable identities between multi-overlap distributions that has been discovered first by Ghirlanda and Guerra in the context of SK models. If $\mu_{\beta,N}^{\otimes n}$ denotes the n-fold product Gibbs measure, the Ghirlanda-Guerra relations assert a recursion relation of the form

$$\begin{aligned}
&\mathbb{E}\mu_{\beta,N}^{\otimes n+1}\left(D_N(\sigma^{n+1},\sigma^k)\leq t|\mathcal{B}_n\right)\\
&=\frac{1}{n}\sum_{\ell\neq k}\mathbb{E}\mu_{\beta,N}^{\otimes n}\left(D_N(\sigma^\ell,\sigma^k)\leq t|\mathcal{B}_n\right)\\
&+\frac{1}{n}\mathbb{E}\mu_{\beta,N}^{\otimes 2}\left(D_N(\sigma^1,\sigma^2)\leq t|\mathcal{B}_n\right)+o(1)
\end{aligned}$$

These relations hold generically for Gaussian mean field models, with D_N being the distance through which the covariance is defined. The proof of these relations is based on Gaussian integration by parts formulas, and concentration of measure inequalities. In the case of the GREM models, where D_N is ultrametric, these recursions are sufficient to determine all n-replica 'overlap' distributions in terms of the 2-replica distribution. On the other hand, the set of n-replica overlap distributions determines the law of the process $\mathcal{K}$ and thus the geometry of the Gibbs measure. In particular, they leave time changes of Neveu's process as only candidates for limit processes. In the case of the SK models, the same does not hold a priori, since the Hamming distance is not an ultrametric. However, since the Parisi solution is correct, this suggests very strongly that asymptotically the overlap distances are almost surely (with respect to the Gibbs measure) ultrametric. Then, the Ghirlanda-Guerra identities also imply that the geometry of the Gibbs measures are described by the same structure.

From mean field to lattice models. One of the widely discussed issues in the theory of spin glasses is to what extent the results of mean field theory are relevant for lattice models. This issue will be addressed in another article (by Newman and Stein). Here we will only mention a recent result of S. Franz and F.L. Toninelli that shows that the free energy of the SK model can be represented as the limit of the free energy of lattice models when the range of the interaction tends to zero while their strength tends to zero in an appropriate way (so called Kac models). This still leaves open many finer questions, but hints to the fact that mean field theory bears at least some relevance for realistic spin glasses.

See also Statistical mechanics, Spin glasses, Dynamics of disordered systems.

References

[1] Aizenman M, Sims R, Starr SL (2003) An Extended Variational Principle for the SK Spin-Glass Model. Phys. Rev. B 6821: 4403

[2] Amit DJ (1989) Modelling Brain Function, Cambridge University Press, Cambridge.

[3] Bovier A (2001) Statistical Mechanics of Disordered Systems, MaPhySto Lecture Notes 10, Aarhus University.

[4] Bovier A, Picco P (1998) (Eds.), Mathematical Aspects of Spin Glasses and Neural Networks, Progress in Probability **41**, Birkhäuser, Boston-Basel-Berlin.

[5] Ghirlanda S, Guerra F (1998) General properties of overlap probability distributions in disordered spin systems. Towards Parisi ultrametricity. J. Phys. A 31, 9149–9155

[6] Guerra F (2002) Broken replica symmetry bounds in the mean field spin glass model. Comm. Math. Phys. 233 (2003), 1–12.

[7] Guerra F, Toninelli, FL (2002) The thermodynamic limit in mean field spin glass models. Comm. Math. Phys. 23, 71–79

[8] Mézard , Parisi G, Virasoro MA (1988) Spin Glass Theory and Beyond, World Scientific, Singapore.

[9] Ruelle D (1987) A mathematical reformulation of Derrida's REM and GREM. Comm. Math. Phys. 108, no. 2, 225–239.

[10] Talagrand M (2003) Spin Glasses: A Challenge for Mathematicians, Ergebnisse der Mathematik und ihrer Grenzgebiete 46, Springer, Berlin-Heidelberg-New York.

[11] Talagrand M (2003) The Parisi formula. To appear in Ann. Math. (2005).

[12] Franz, S, Toninelli, FL (2004) Finite-range spin glasses in the Kac limit: free energy and local observables. J. Phys. A 37, 7433–7446.