# Rigorous results on some simple spin glass models

**Anton Bovier**[1,2]
*Weierstraß–Institut*
*für Angewandte Analysis und Stochastik*
*Mohrenstrasse 39, D-10117 Berlin, Germany*

**Irina Kurkova**[3]
*Laboratoire de Probabilités et Modèles Aléatoires*
*Université Paris 6*
*4, place Jussieu, B.C. 188*
*75252 Paris, Cedex 5, France*

**Abstract:** In this paper we review some recent rigorous results that provide an essentially complete solution of a class of spin glass models introduced by Derrida in the 1980ies. These models are based on Gaussian random processes on $\{-1,1\}^N$ whose covariance is a function of a ultrametric distance on that set. We prove the convergence of the free energy as well as the Gibbs measures in an appropriate sense. These results confirm fully the predictions of the replica method including in situations where continuous replica symmetry breaking takes place.

*Keywords:* Gaussian processes,generalized random energy model, continuous hierarchies, spin glasses, Poisson cascades, probability cascades, Ghirlanda-Guerra identities.

*AMS Subject Classification:* 82B44, 60G70,60K35

[1] e-mail: bovier@wias-berlin.de
[2] Research supported in part by the DFG in the Concentration program "Interacting stochastic systems of high complexity".
[3] e-mail: kourkova@ccr.jussieu.fr

## 1. Introduction.

In spite of considerable recent progress [T1,T3,T4,GT,G], there remains a considerable gap between the heuristic understanding of mean field spin glasses such as the Sherrington-Kirkpatrick model [SK] (see [MPV]) , and the mathematical understanding of the properties of such models. We have now a reasonably good insight in situations when the so-called "replica symmetric" solution is expected to hold, but already solution of a model with one-step replica symmetry breaking has required an enormous effort [T6]. Understanding situations with full continuous replica symmetry breaking in the context of SK models appears presently quite hopeless, even though Guerra [G] has proven very recently an extremely interesting result that shows that in the standard SK model, Parisi's solution provides a lower bound for the free energy.

In this note we report on progress in understanding the emergence of replica symmetry breaking in the context of a class of "simple" spin glass models, introduced by Derrida in 1980: the *random energy model* (REM)[D1,D2], and the *generalised random energy model*(GREM)[D3,DG1,DG2,DG3]. The former consisted of modelling the random energy landscape as simply i.i.d. Gaussian random variables on the set of spin configurations, $\{-1,1\}^N$. This model can be seen formerly as the limit of the so-called $p$-spin SK-models [SK], when $p$ tends to infinity [D1]. In spite of its simplicity, this model has proven to be a rather instructive toy model, and has received considerable attention in the mathematical community [DG3,DW,Ei,OP,GMP,Ru,BKL,KP, B]. Of course, in many respects this model is mathematically almost trivial, and physically quite unrealistic, as all the dependence structure that is present in more realistic models like the SK model, is absent. The GREM was introduced in view of *keeping* dependence, while simplifying it to a *hierarchical* structure to still yield a mathematically more tractable model. In fact, the GREM can be seen as a class of models that is obtained by equipping the hypercube $\{-1,1\}^N$ with a tree structure and an associated ultra-metric distance, and then considering standardized Gaussian random fields on the hypercube whose correlation function depends only on this distance. We will call these models *"Derrida's models"* in contrast to the *"Sherrington/Kirkpatrick (SK) models"* where the covariance depends on the Hamming distance, respectively the overlap $R_N(\sigma,\sigma') = N^{-1}\sum_{i=1}^N \sigma_i\sigma_j$.

In [DG1], B. Derrida and E. Gardner presented a solution of the model with finitely many hierarchies in the sense that they computed the free energy in the thermodynamic limit. A rigorous derivation of this solution (in a somewhat more elegant form) was later obtained by

Cappocaccia et al. [CCP]. Derrida and Gardner also considered the limit of their formulae when the number of hierarchies tends to infinity. They argued that for suitable choices of the covariance function, this limits yield approximations for the standard $p$-spin SK models, even though, as they point out, the quality of the approximations is not spectacular.

In this paper we review recent results obtained in [BKL,B,BK1,BK2,BK3] that give an essentially complete solution confirming the results of the replica method for all these models. In Section 2 we present first in detail the rather simple case of the REM which will serve as a pedagogical example. In Section 3 we then turn to the general class of Derrida's models.

## 2. The random energy model.

The random energy model, introduced by Derrida [D1,D2] can be considered as the ultimate toy model of a disordered system. In this model, rather little is left of the structure of interacting spins, but we will still be able to gain a lot of insight into the peculiarities of disordered systems by studying this simple system. For rigorous work on the REM see e.g. [Ei,OP,GMP,DW,BKL,T5].

The REM is a model with state space $\mathcal{S}_N = \{-1,+1\}^N$. For fixed $N$, the Hamiltonian is given by

$$H_N(\sigma) = -\sqrt{N} X_\sigma \tag{2.1}$$

where $X_\sigma$, is a family of $2^N$ i.i.d. centered normal random variables.

### 2.1. The free energy.

Before turning to the question of Gibbs measures, we turn to the simpler question of analysing in some detail the partition function. In this model, the partition function is of course just the sum of i.i.d. random variables, i.e.

$$Z_{\beta,N} \equiv 2^{-N} \sum_{\sigma\in\mathcal{S}_N} e^{\beta\sqrt{N}X_\sigma} \tag{2.2}$$

One usually asks first for the exponential asymptotics of this quantity, i.e. one introduces the *free energy*,

$$F_{\beta,N} \equiv -\frac{1}{N} \ln Z_{\beta,N} \tag{2.3}$$

and tries to find its limit as $N \uparrow \infty$. Let me mention that in general mean field spin glasses, the existence of the limit even of the averaged free energy has been a long standing open problem. While writing this note, a preprint by Guerra and Toninelli [GT] has appeared in which a simple and clever proof of the existence of the limit in a rather large class of mean field spin glass models is given.

In this simple model one can compute this limit exactly. In fact it was found be Derrida [D1] that:

***Theorem 2.1:*** *In the REM,*

$$\lim_{N\uparrow\infty} \mathbb{E}F_{\beta,N} = \begin{cases} -\frac{\beta^2}{2}, & \text{for } \beta \leq \beta_c \\ -\frac{\beta_c^2}{2} - (\beta-\beta_c)\beta_c, & \text{for } \beta \geq \beta_c \end{cases} \tag{2.4}$$

*where* $\beta_c = \sqrt{2\ln 2}$.

**2.2. Fluctuations and limit theorems.**

Knowing the free energy is important, but, as one may expect, it is not enough to understand the properties of the Gibbs measures completely. It is the analysis of the fluctuations of the free energy that will reveal, as we will see, the necessary information. In the REM this can be done using classical results from the theory of extreme value statistics. The proofs are, nonetheless, quite cumbersome, and may be found in in [BKL] or [B].

***Theorem 2.2:*** *The partition function of the REM has the following fluctuations:*

*(i) If* $\beta < \sqrt{\ln 2/2}$, *then*

$$e^{\frac{N}{2}(\ln 2-\beta^2)} \ln \frac{Z_{\beta,N}}{\mathbb{E}Z_{\beta,N}} \stackrel{\mathcal{D}}{\to} \mathcal{N}(0,1). \tag{2.5}$$

*(ii) If* $\beta = \sqrt{\ln 2/2}$, *then*

$$\sqrt{2}e^{\frac{N}{2}(\ln 2-\beta^2)} \ln \frac{Z_{\beta,N}}{\mathbb{E}Z_{\beta,N}} \stackrel{\mathcal{D}}{\to} \mathcal{N}(0,1). \tag{2.6}$$

*(iii) Let* $\alpha \equiv \beta/\sqrt{2\ln 2}$. *If* $\sqrt{\ln 2/2} < \beta < \sqrt{2\ln 2}$, *then*

$$e^{\frac{N}{2}(\sqrt{2\ln 2}-\beta)^2+\frac{\alpha}{2}[\ln(N\ln 2)+\ln 4\pi]} \ln \frac{Z_{\beta,N}}{\mathbb{E}Z_{\beta,N}} \stackrel{\mathcal{D}}{\to} \int_{-\infty}^{\infty} e^{\alpha z}(\mathcal{P}(dz) - e^{-z}dz), \tag{2.7}$$

*where* $\mathcal{P}$ *denotes the Poisson point process*[4] *on* $\mathbb{R}$ *with intensity measure* $e^{-x}dx$.

*(iv) If* $\beta = \sqrt{2\ln 2}$, *then*

$$e^{\frac{1}{2}[\ln(N\ln 2)+\ln 4\pi]}\left(\frac{Z_{\beta,N}}{\mathbb{E}Z_{\beta,N}} - \frac{1}{2} + \frac{\ln(N\ln 2)+\ln 4\pi}{4\sqrt{\pi N\ln 2}}\right) \stackrel{\mathcal{D}}{\to} \int_{-\infty}^{0} e^{z}(\mathcal{P}(dz) - e^{-z}dz) + \int_0^{\infty} e^z\mathcal{P}(dz). \tag{2.8}$$

[4]For a thorough exposition on point processes and their connection to extreme value theory, see in particular [Re].

*(v) If $\beta > \sqrt{2\ln 2}$, then*

$$e^{-N[\beta\sqrt{2\ln 2}-\ln 2]+\frac{\alpha}{2}[\ln(N\ln 2)+\ln 4\pi]} Z_{\beta,N} \xrightarrow{\mathcal{D}} \int_{-\infty}^{\infty} e^{\alpha z} \mathcal{P}(dz) \tag{2.9}$$

*and*

$$\ln Z_{\beta,N} - \mathbb{E}\ln Z_{\beta,N} \xrightarrow{\mathcal{D}} \ln \int_{-\infty}^{\infty} e^{\alpha z}\mathcal{P}(dz) - \mathbb{E}\ln \int_{-\infty}^{\infty} e^{\alpha z}\mathcal{P}(dz). \tag{2.10}$$

**Remark:** Note that expressions like $\int_{-\infty}^{0} e^z(\mathcal{P}(dz) - e^{-z}dz)$ are always understood as $\lim_{y\downarrow -\infty} \int_y^0 e^z(\mathcal{P}(dz) - e^{-z}dz)$. All the functionals of the Poisson point process appearing are almost surely finite random variables. Note that the limit in (2.7) has infinite variance and the one in (2.9) has infinite mean.

Let us just briefly comment on how these results are obtained. In fact, (i) follows from the standard CLT for arrays of independent random variables under Lindeberg's condition.

As the Lindeberg condition fails for $2\beta^2 \geq \ln 2$, it is clear that we cannot expect a simple CLT beyond this regime. Such a failure of a CLT is always a problem related to "heavy tails", and results from the fact that extremal events begin to influence the fluctuations of the sum. It appears therefore reasonable to separate from the sum the terms where $X_\sigma$ is anomalously large. For Gaussian r.v.'s it is well known that the right scale of separation is given by $u_N(x)$ defined by

$$2^N \int_{u_N(x)}^{\infty} \frac{dz}{\sqrt{2\pi}} e^{-z^2/2} = e^{-x} \tag{2.11}$$

which (for $x > -\ln N/\ln 2$) is equal to (see e.g. [LLR])

$$u_N(x) = \sqrt{2N\ln 2} + \frac{x}{\sqrt{2N\ln 2}} - \frac{\ln(N\ln 2) + \ln 4\pi}{2\sqrt{2N\ln 2}} + o(1/\sqrt{N}), \tag{2.12}$$

$x \in \mathbb{R}$ is a parameter. The key to most of what follows relies on the famous result on the convergence of the extreme value process to a Poisson point process. Let us now introduce the point process on $\mathbb{R}$ given by

$$\mathcal{P}_N \equiv \sum_{\sigma\in\mathcal{S}_N} \delta_{u_N^{-1}(X_\sigma)}. \tag{2.13}$$

A classical result from the theory of extreme order statistics (see e.g. [LLR]) asserts that

***Theorem 2.3:*** *The point process $\mathcal{P}_N$ converges weakly to a Poisson point process on $\mathbb{R}$ with intensity measure $e^{-x}dx$.*

The key idea is then to split the sum by a cutoff corresponding to whether $X_\sigma$ is bigger or smaller than $u_N(x)$; the former can then be represented as a functional of the extremal process that converges to the Poisson process, and the latter has to be controlled carefully. The computations are in fact quite tedious.

If we write

$$Z_{\beta,N} = Z^x_{\beta,N} + (Z_{\beta,N} - Z^x_{\beta,N}) \tag{2.14}$$

for $\beta \geq \sqrt{2\ln 2}$

$$Z_{\beta,N} - Z^x_{\beta,N} = e^{N\left[\beta\sqrt{2\ln 2}-\ln 2\right]-\frac{\alpha}{2}[\ln(N\ln 2)+\ln 4\pi]} \sum_{\sigma\in\mathcal{S}_N} \mathbb{1}_{\{u_N^{-1}(\sigma)>x\}} e^{\alpha u_N^{-1}(X_\sigma)} \tag{2.15}$$

so that for any $x \in \mathbb{R}$,

$$(Z_{\beta,N} - Z^x_{\beta,N})e^{-N\left[\beta\sqrt{2\ln 2}-\ln 2\right]+\frac{\alpha}{2}[\ln(N\ln 2)+\ln 4\pi]} \xrightarrow{\mathcal{D}} \int_x^\infty e^{\alpha z}\mathcal{P}(dz). \tag{2.16}$$

The remaining term is shown to converge to zero in probability as first $N \uparrow \infty$ and then $x \downarrow -\infty$. $\diamondsuit$

### 2.3. The Gibbs measure.

With our preparation on the fluctuations of the free energy, we have accumulated enough understanding about the partition function that we can deal with the Gibbs measures. Clearly, there are a number of ways of trying to describe the asymptotics of the Gibbs measures. We are seeking a result on the convergence in distribution of random measures. To be able to state such a results, we have to introduce a topology on the spin configuration state that makes it uniformly compact. The usual topology to do this would be product topology, and this clearly would be an option here. However, given what we already know about the partition function, this topology does not appear suited to give describe the measure appropriately. Part (v) of Theorem 2.2 actually implies that the partition function is dominated by a 'few' spin configurations with exceptionally large energy. This is a feature that should remain visible in a limit theorem. The question we therefore must address in mean field models is how to describe a limiting measure on an infinite dimensional cube that properly reflects the symmetry (under permutation) of the finite dimensional object, in other words that views this object in an unbiased way.

A first attempt consists in mapping the hypercube to the interval $[-1, 1]$ via

$$\mathcal{S}_N \ni \sigma \to r_N(\sigma) \equiv \sum_{i=1}^{N} \sigma_i 2^{-i} \in [-1, 1] \tag{2.17}$$

Define the pure point measure $\tilde{\mu}_{\beta,N}$ on $[-1,1]$ by

$$\tilde{\mu}_{\beta,N} \equiv \sum_{\sigma \in \mathcal{S}_N} \delta_{r_N(\sigma)} \mu_{\beta,N}(\sigma) \tag{2.18}$$

Our results will be expressed in terms of the convergence of these measures. It will be understood in the sequel that the space of measures on $[-1,1]$ is equipped with the topology of weak convergence, and all convergence results hold with respect to this topology.

As the diligent reader will have expected, in the high temperature phase the limit is the same as for $\beta = 0$, namely

***Theorem 2.4:*** *If $\beta < \sqrt{2 \ln 2}$, then*

$$\tilde{\mu}_{\beta,N} \to \frac{1}{2}\lambda, \quad a.s. \tag{2.19}$$

*where $\lambda$ denotes the Lebesgue measure on $[-1,1]$.*

***Proof:*** Note that we have to prove that for any finite collection of intervals $I_1, \ldots, I_k \subset [-1,1]$, the family of random variables $\{\tilde{\mu}_{\beta,N}(I_1), \ldots, \tilde{\mu}_{\beta,N}(I_k)\}$ converges jointly almost surely to $\frac{1}{2}|I_1|, \ldots, \frac{1}{2}|I_k|$. But by construction these random vectors are independent, so that this will follow automatically, if we can prove the result in the case $k = 1$. Our strategy is to get first very sharp estimates for a family of special intervals.

In the sequel we will always assume that $N \geq n$. We will denote by $\Pi_n$ the canonical projection from $\mathcal{S}_N$ to $\mathcal{S}_n$. To simplify notation, we will often write $\sigma_n \equiv \Pi_n \sigma$ when no confusion can arise. For $\sigma \in \mathcal{S}_N$, set

$$a_n(\sigma) \equiv r_n(\Pi_n \sigma) \tag{2.20}$$

and

$$I_n(\sigma) \equiv [a_n(\sigma) - 2^{-n}, a_n(\sigma) + 2^{-n}) \tag{2.21}$$

Note that the union of all these intervals forms a disjoint covering of $[-1,1)$. Obviously, these intervals are constructed in such a way that

$$\tilde{\mu}_{\beta,N}(I_n(\sigma)) = \mu_{\beta,N}\left(\{\sigma' \in \mathcal{S}_N : \Pi_n(\sigma') = \Pi_n(\sigma)\}\right) \tag{2.22}$$

The first step in the proof consists in showing that the masses of all the intervals $I_n(\sigma)$ are remarkably well approximated by their uniform mass.

**Lemma 2.5:** *Set* $\beta' \equiv \sqrt{\frac{N}{N-n}}\beta$*. For any* $\sigma \in \mathcal{S}_n$*,*

*(i) If* $\beta' \leq \sqrt{\frac{\ln 2}{2}}$*,*

$$|\tilde{\mu}_{\beta,N}(I_n(\sigma)) - 2^{-n}| \leq 2^{-n} e^{-(N-n)(\ln 2 - \beta'^2)} Y_{N-n} \tag{2.23}$$

*where* $Y_N$ *has bounded variance, as* $N \uparrow \infty$*.*

*(ii) If* $\sqrt{\frac{\ln 2}{2}} < \beta' < \sqrt{2\ln 2}$*,*

$$|\tilde{\mu}_{\beta,N}(I_n(\sigma)) - 2^{-n}| \leq 2^{-n} e^{-(N-n)(\sqrt{2\ln 2} - \beta')^2/2 - \alpha \ln(N-n)/2} Y_{N-n} \tag{2.24}$$

*where* $Y_N$ *is a random variable with bounded mean modulus.*

*(iii) If* $\beta = \sqrt{2\ln 2}$*, then, for any* $n$ *fixed,*

$$|\tilde{\mu}_{\beta,N}(I_n(\sigma)) - 2^{-n}| \to 0 \quad \textit{in probability} \tag{2.25}$$

**Remark:** Note that in the sub-critical case, the results imply convergence to the uniform product measure on $\mathcal{S}$ in a *very strong sense.* In particular, the base-size of the cylinders considered (i.e. $n$) can grow proportionally to $N$, *even if almost sure convergence uniformly for all cylinders is required!* This is unusually good. However, one should not be deceived by this fact: even though seen from the cylinder masses the Gibbs measures look like the uniform measure, seen from the point of view of individual spin configurations the picture is quite different. In fact, the measure concentrates on an *exponentially* small fraction of the full hypercube, namely those $O(\exp(N(\ln 2 - \beta^2/2)))$ vertices that have energy $\sim \beta N$ (Exercise!). It is just the fact that this set is still exponentially large, as long as $\beta < \sqrt{2\ln 2}$, and is very uniformly dispersed over $\mathcal{S}_N$, that produces this somewhat paradoxical effect. The rather weak result in the critical case is not artificial. In fact it is not true that almost sure convergence will hold. This follows e.g. from Theorem 1 in [GMP]. One should of course anticipate some signature of the phase transition at the critical point.

***Proof:*** The proof of this lemma is a simple application of the first three points in Theorem 2.2. Just note that the partial partition functions

$$Z_{\beta,N}(\sigma_n) \equiv \mathbb{E}_{\sigma'} e^{\beta\sqrt{N}X_{\sigma'}} \mathbb{I}_{\Pi_n(\sigma')=\sigma_n} \tag{2.26}$$

are independent and have the same distribution as $2^{-n} Z_{\beta',N-n}$. But

$$\tilde{\mu}_{\beta,N}(I_n(\sigma_n)) = \frac{Z_{\beta,N}(\sigma_n)}{[Z_{\beta,N} - Z_{\beta,N}(\sigma_n)] + Z_{\beta,N}(\sigma_n)} \tag{2.27}$$

Note that $Z_{\beta,N}(\sigma_n)$ and $[Z_{\beta,N} - Z_{\beta,N}(\sigma_n)]$ are independent. It should now be obvious how to conclude the proof with the help of Theorem 2.2. $\diamondsuit$

Once we have the excellent approximation of the measure on all of the intervals $I_n(\sigma)$, almost sure convergence of the measure in the weak topology is a simple consequence. Of course, this is just a coarse version of the finer results we have, and much more precise information on the quality of approximation can be inferred from Lemma 2.5. But since the high-temperature phase is not our prime concern, we will not go further in this direction.

Somehow much more interesting is the behaviour of the measure at low temperatures that we will discuss now. Let us introduce the Poisson point process $\mathcal{R}$ on the strip $[-1,1] \times \mathbb{R}$ with intensity measure $\frac{1}{2}dy \times e^{-x}dx$. If $(Y_k, X_k)$ denote the atoms of this process, define a new point process $\mathcal{W}_\alpha$ on $[-1,1] \times (0,1]$ whose atoms are $(Y_k, w_k)$, where

$$w_k \equiv \frac{e^{\alpha X_k}}{\int \mathcal{R}(dy,dx)e^{\alpha x}} \tag{2.28}$$

for $\alpha > 1$. Let us note that the process $\widehat{\mathcal{W}} = \sum_k w_k$ is known in the literature as the *Poisson-Dirichlet process* with parameter $\alpha$ [K].

With this notation we have that

***Theorem 2.6:*** *If $\beta > \sqrt{2\ln 2}$, with $\alpha = \beta/\sqrt{2\ln 2}$,*

$$\tilde{\mu}_{\beta,N} \xrightarrow{\mathcal{D}} \tilde{\mu}_\beta \equiv \int_{[-1,1]\times(0,1]} \mathcal{W}_\alpha(dy,dw)\delta_y w \tag{2.29}$$

***Proof:*** With $u_N(x)$ defined in (2.12), we define the point process $\mathcal{R}_N$ on $[-1,1] \times \mathbb{R}$ by

$$\mathcal{R}_N \equiv \sum_{\sigma\in\mathcal{S}_N} \delta_{(r_N(\sigma), u_N^{-1}(X_\sigma))} \tag{2.30}$$

A standard result of extreme value theory (see [LLR], Theorem 5.7.2) is easily adapted to yield that

$$\mathcal{R}_N \xrightarrow{\mathcal{D}} \mathcal{R}, \quad \text{as } N \uparrow \infty \tag{2.31}$$

where the convergence is in the sense of weak convergence on the space of sigma-finite measures endowed with the (metrizable) topology of vague convergence. Note that

$$\mu_{\beta,N}(\sigma) = \frac{e^{\alpha u_N^{-1}(X_\sigma)}}{\sum_\sigma e^{\alpha u_N^{-1}(X_\sigma)}} = \frac{e^{\alpha u_N^{-1}(X_\sigma)}}{\int \mathcal{R}_N(dy,dx)e^{\alpha x}} \tag{2.32}$$

Since $\int \mathcal{R}_N(dy,dx)e^{\alpha x} < \infty$ a.s., we can define the point process

$$\mathcal{W}_N \equiv \sum_{\sigma\in\mathcal{S}_N} \delta_{\left(r_N(\sigma), \frac{\exp(\alpha u_N^{-1}(X_\sigma))}{\int \mathcal{R}_N(dy,dx)\exp(\alpha x)}\right)} \tag{2.33}$$

on $[-1,1]\times(0,1]$. Then

$$\tilde{\mu}_{\beta,N} = \int \mathcal{W}_N(dy,dw)\delta_y w \tag{2.34}$$

The only non-trivial point in the convergence proof is to show that the the contribution to the partition functions in the denominator from atoms with $u_N(X_\sigma) < x$ vanishes as $x \downarrow -\infty$. But this is precisely what we have shown to be the case in the proof of part (v) of Theorem 2.2. Standard arguments then imply that first $\mathcal{W}_N \xrightarrow{\mathcal{D}} \mathcal{W}$, and consequently, (2.29). $\diamondsuit$

**Remark:** Note that Theorem 2.6 contains in particular the convergence of the Gibbs measure in the product topology on $\mathcal{S}_N$, since cylinders correspond to certain subintervals of $[-1,1]$. On the other hand, it implies that the point process of weights $\sum_{\sigma\in\mathcal{S}_N}\delta_{\mu_{\beta,N}(\sigma)}$ converges in law to the marginal of $\mathcal{W}_N$ on $(0,1]$ which is the process introduced by Ruelle [Ru]. The formulation of Theorem 2.6 is moreover very much in the spirit of the meta-state approach to random Gibbs measures [NS]. The limiting measure is a measure on a continuous space, and each point measure on this set may appear as "pure state". The "meta-state", i.e. the law of the random measure $\tilde{\mu}_\beta$ is a probability distribution concentrated on the countable convex combinations of pure states randomly chosen by a Poisson point process from an uncountable collection, while the coefficients of the convex combination are again random and selected via another point process.

Let us discuss the properties of the limiting measure $\tilde{\mu}_\beta$. It is not hard to see that with probability one, the support of $\tilde{\mu}_\beta$ is the entire interval $[-1,1]$. On the other hand, its mass is concentrated on a countable set, i.e. the measure is pure point. To see this, consider the rectangle $A_\epsilon \equiv (\ln\epsilon,\infty)\times[-1,1]$. Clearly, the process $\mathcal{R}$ restricted to this set has finite total intensity given by $\epsilon^{-1}$. i.e. the number total number of atoms in that set is a Poissonian random variable with parameter $\epsilon^{-1}$. Now if we remove the projection of these finitely many random points from $[-1,1]$, we will show that the total mass that remains goes to zero with $\epsilon$. Clearly, the remaining mass is given by

$$\int_{[-1,1]\times(-\infty,\ln\epsilon)} \mathcal{R}(dy,dx)\frac{e^{\alpha x}}{\int \mathcal{P}(dx')e^{\alpha x'}} = \int_{-\infty}^{\ln\epsilon} \mathcal{P}(dx)\frac{e^{\alpha x}}{\int \mathcal{P}(dx')e^{\alpha x'}} \tag{2.35}$$

We want to get a lower bound in probability on the denominator. The simplest possible bound is obtained by estimating the probability of the integral by the contribution of the largest atom which of course follows the double-exponential distribution. Thus

$$\mathbb{P}\left[\int \mathcal{P}(dx)e^{\alpha x} \leq Z\right] \leq e^{-e^{-\ln Z/\alpha}} = e^{-Z^{-\frac{1}{\alpha}}} \tag{2.36}$$

Setting $\Omega_Z \equiv \{\mathcal{P} : \int \mathcal{P}(dx)e^{\alpha x} \leq Z\}$, we conclude that, for $\alpha > 1$,

$$\begin{aligned}
\mathbb{P}\left[\int_{-\infty}^{\ln \epsilon} \mathcal{P}(dx)\frac{e^{\alpha x}}{\int \mathcal{P}(dx')e^{\alpha x'}} > \gamma\right] &\leq \mathbb{P}\left[\int_{-\infty}^{\ln \epsilon} \mathcal{P}(dx)\frac{e^{\alpha x}}{\int \mathcal{P}(dx')e^{\alpha x'}} > \gamma,\, \Omega_Z^c\right] + \mathbb{P}[\Omega_Z]\\
&\leq \mathbb{P}\left[\int_{-\infty}^{\ln \epsilon} \mathcal{P}(dx)e^{\alpha x} > \gamma Z,\, \Omega_Z^c\right] + \mathbb{P}[\Omega_Z]\\
&\leq \mathbb{P}\left[\int_{-\infty}^{\ln \epsilon} \mathcal{P}(dx)e^{\alpha x} > \gamma Z\right] + \mathbb{P}[\Omega_Z]\\
&\leq \frac{\mathbb{E}\int_{-\infty}^{\ln \epsilon} \mathcal{P}(dx)e^{\alpha x}}{\gamma} + \mathbb{P}[\Omega_Z]\\
&\leq \frac{\epsilon^{\alpha-1}}{(\alpha-1)\gamma Z} + e^{-Z^{-\frac{1}{\alpha}}}
\end{aligned} \tag{2.37}$$

Obviously, for any positive $\gamma$ it is possible to choose $Z$ as a function of $\epsilon$ in such a way that the right hand side tends to zero. But this implies that with probability one, all of the mass of the measure $\tilde{\mu}_\beta$ is carried by a countable set, implying that $\tilde{\mu}_\beta$ is pure point.

So we see that the phase transition in the REM expresses itself via a change of the properties of the infinite volume Gibbs measure mapped to the interval from Lebesgue measure at high temperatures to a random dense pure point measure at low temperatures.

**2.4. The replica overlap.**

While the random measure description of the phase transition in the REM appears rather nice, one would argue that it ignores fully the geometry of the state space as a hypercube. A neat object to measure look at in this respect would be the mass distribution around a given configuration,

$$m_\sigma(t) \equiv \mu_{\beta,N}\left(R_N(\sigma,\sigma') \geq t\right) \tag{2.38}$$

where the $\sigma$ is fixed and the measure $\mu$ refers to the configuration $\sigma'$. $m_\sigma(\cdot)$ is a probability distribution function on $[-1,1]$. As a function of $\sigma$, this is a measure values random variable. Taking the overage of this quantity again with respect to the Gibbs distribution of $\sigma$, we obtain the popular "overlap distribution",

$$f_{\beta,N}[\omega](dz) \equiv \mu_{\beta,N}\left(m_\sigma(dz)\right) = \mu_{\beta,N}[\omega] \otimes \mu_{\beta,N}[\omega]\left(R_N(\sigma,\sigma') \in dz\right) \tag{2.39}$$

It turns out that a much richer object is obtained by passing to a measure valued quantity, namely

$$\mathcal{K}_{\beta,N} \equiv \sum_{\sigma\in\mathcal{S}_N} \mu_{\beta,N}(\sigma)\delta_{m_\sigma(\cdot)} \tag{2.40}$$

This measure tells us the probability to see a given miss distribution around oneself, if one is distributed with the Gibbs measure. Of course we have that

$$f_{\beta,N}[\omega](\cdot) = \int \mathcal{K}_{\beta,N}(dm)m(\cdot) \tag{2.41}$$

Of course, in the REM, one is not likely to see anything very exciting, the overlap distribution is asymptotically concentrated on the values 0 and 1 only:

***Theorem 2.7:***

*(i) For all $\beta < \sqrt{2\ln 2}$*

$$\lim_{N\uparrow\infty} f_{\beta,N} = \delta_0, \quad a.s. \tag{2.42}$$

*(ii) For all $\beta > \sqrt{2\ln 2}$*

$$f_{\beta,N} \stackrel{\mathcal{D}}{\to} \delta_0 \left(1 - \int \mathcal{W}(dy,dw)w^2\right) + \delta_1 \int \mathcal{W}(dy,dw)w^2 \tag{2.43}$$

*(iii) The random measures $\mathcal{K}_{\beta,N}$ converge to a random probability distribution $\mathcal{K}_\beta$ that is supported on the atomic measures with support on $\{0,1\}$, more precisely if $\beta > \sqrt{2\ln 2}$,*

$$\mathcal{K}_\beta = \int \mathcal{W}(dy,dw)w\delta_{w\delta_1+(1-w)\delta_0} \tag{2.44}$$

*while for $\beta < \sqrt{2\ln 2}$, $\mathcal{K}_\beta$ is the Dirac mass on the Dirac mass concentrated at* 0.

***Proof:*** We will write for any $I \subset [-1,1]$

$$f_{\beta,N}(I) = Z_{\beta,N}^{-2}\mathbb{E}_\sigma\mathbb{E}_{\sigma'} \sum_{\substack{t\in I\\ R_N(\sigma,\sigma')=t}} e^{\beta\sqrt{N}(X_\sigma+X_{\sigma'})} \tag{2.45}$$

First of all, the denominator is bounded from below by $[\widetilde{Z}_{\beta,N}(c)]^2$, and, with probability of order $\delta^{-2}\exp(-Ng(c,\beta))$, this in turn is larger than $(1-\delta)^2[\mathbb{E}\widetilde{Z}_{\beta,N}(c)]^2$. Now let first $\beta < \sqrt{2\ln 2}$. Assume first that $I \subset (0,1)\cup[-1,0)$. We conclude that

$$\begin{aligned}\mathbb{E}f_{\beta,N}(I) &\leq \frac{1}{(1-\delta)^2}\mathbb{E}_\sigma\mathbb{E}_{\sigma'} \sum_{\substack{t\in I\\ R_N(\sigma,\sigma')=t}} 1 + \delta^{-2}e^{-g(c,\beta)N}\\ &= \frac{1}{\sqrt{2\pi N}}\frac{1}{(1-\delta)^2}\sum_{t\in I}\frac{2e^{-N\phi(t)}}{1-t^2} + \delta^{-2}e^{-g(c,\beta)N}\end{aligned} \tag{2.46}$$

for any $\beta < c < \sqrt{2\ln 2}$, where $\phi : [-1,1] \to \mathbb{R}$ denotes the Cramèr entropy function

$$\phi(t) = \frac{(1+t)}{2}\ln(1+t) + \frac{(1-t)}{2}\ln(1-t) \tag{2.47}$$

Here we used of course that, firstly, if $(1-t)N = 2\ell$, $\ell = 0,\dots,N$, then

$$\mathbb{E}_\sigma \mathbb{E}_{\sigma'} \mathbb{1}_{R_N(\sigma,\sigma')=t} = 2^{-N}\binom{N}{\ell} \tag{2.48}$$

and, secondly, Stirling's approximation which implies that

$$\binom{N}{\ell} = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{N}{\ell(N-\ell)}}\frac{N^N}{\ell^\ell(N-\ell)^{N-\ell}}(1+o(1)) \tag{2.49}$$

valid if $\ell \sim xN$ with $x \in (0,1)$. Under our assumptions on $I$, we see immediately from this representation that the right hand side of (2.46) is clearly exponentially small in $N$. If $1 \in I$, the additional term coming from $t=1$ gives an exponentially small contribution. This shows that the measure $f_{\beta,N}$ concentrates asymptotically on the point 0. This proves (2.42).

Now let $\beta > \sqrt{2\ln 2}$. Here we use the sharper truncations introduced in 2.2. Note first that for any interval $I$

$$\left| f_{\beta,N}(I) - Z_{\beta,N}^{-2}\mathbb{E}_\sigma\mathbb{E}_{\sigma'}\sum_{\substack{t\in I\\ R_N(\sigma,\sigma')=t}} \mathbb{1}_{X_\sigma,X_{\sigma'}\geq u_N(x)} e^{\beta\sqrt{N}(X_\sigma+X_{\sigma'})}\right| \leq \frac{2Z_{\beta,N}^x}{Z_{\beta,N}} \tag{2.50}$$

The proof of Theorem 2.2 shows that the right hand side of (2.50) tends zero in probability as first $N \uparrow \infty$ and then $x \downarrow -\infty$. On the other hand, for $t \neq 1$

$$\begin{aligned}&\mathbb{P}\left[\exists_{\sigma,\sigma,:R_N(\sigma,\sigma')=t} X_\sigma > u_N(x) \wedge X'_\sigma > u_N(x)\right]\\ &\leq \mathbb{E}_\sigma \mathbb{1}_{R_N(\sigma,\sigma')=t}\, 2^{-2N}\mathbb{P}\left[X_\sigma > u_N(x)\right]^2 = \frac{2}{\sqrt{2\pi N}\sqrt{1-t^2}}e^{-\phi(t)N}e^{2x}\end{aligned} \tag{2.51}$$

by the definition of $u_N(x)$ (see (2.11)). This implies again that any interval $I \subset (0,1)\cup[-1,0)$ will have zero mass. To conclude the proof it will be enough to compute $f_{\beta,N}(1)$. Clearly

$$f_{\beta,N}(1) = \frac{2^{-N}\mathbb{Z}_{2\beta,N}}{Z_{\beta,N}^2} \tag{2.52}$$

By Theorem 2.2, (v), one sees easily that

$$f_{\beta,N}(1) \stackrel{\mathcal{D}}{\to} \frac{\int e^{2\alpha z}\mathcal{P}(dz)}{\left(\int e^{\alpha z}\mathcal{P}(dz)\right)^2} \tag{2.53}$$

Expressing the left hand side of (2.53) in terms of the point process $\mathcal{W}_\alpha$ defined in (2.28) yields the expression for the mass of the atom at 1; since the only other atom is at zero the full results follows from the fact that $f_{\beta,N}$ is a probability measure.

The assertions on the measure $\mathcal{K}_{\beta,N}$ are essentially a corollary of the preceding results. The fact that $f_\beta$ is a sum of $\delta_0$ and $\delta_1$ implies immediately that the probability that $m_\sigma$ is not such a sum tends to zero. The explicit formula (2.44) is then quite straightforward. $\diamondsuit$

**2.5. Multi-overlaps and Ghirlanda–Guerra identities.**

It will be interesting to see that the random measures $\mathcal{K}_\beta$ can be controlled with the help of some remarkable algebraic identities that in fact allow us to avoid the detailed analysis of fluctuations performed in Section 2.2.

Let us first note that the convergence of the measures $\mathcal{K}_{\beta,N}$ can be controlled through their moments, which can be written as follows:

$$\begin{aligned}
&\mathbb{E}\left(\int \mathcal{K}_{\beta,N}(dm)m^{k_1}\cdots\int \mathcal{K}_{\beta,N}(dm)m^{k_l}\right)\\
&=\mathbb{E}\mu_{\beta,N}^{\otimes l}\left(m_{\sigma^1}^{k_1}(\cdot)\dots m_{\sigma^l}^{k_l}(\cdots)\right)\\
&=\mathbb{E}\mu_{\beta,N}^{l+k_1+\dots+k_l}\Big(R_N(\sigma^1,\sigma^{l+1})\in\cdot,\dots,R_N(\sigma^1,\sigma^{l+k_1})\in\cdot,\dots,\\
&\qquad\dots,R_N(\sigma^l,\sigma^{l+k_1+\dots+k_{l-1}+1})\in\cdot,\dots,R_N(\sigma^l,\sigma^{l+k_1+\dots+k_l})\in\cdot\Big)
\end{aligned}\tag{2.54}$$

The right hand side is a (marginal of) the distribution of the $m(m-1)$ replica overlaps under the averaged product Gibbs measure on $m=l+k_1+\dots+k_{l-1}+1$ independent replicas of the spin variables. Thus, if we can show that these multi-replica distributions converge, as $N\uparrow\infty$, then the convergence of the measures $\mathcal{K}_{\beta,N}$ will be proven. This is a general fact, which has nothing to do with the particular model we look at. In the REM, of course, considerable simplification will take place since we know that the overlap takes only the values 0 and one in the limit, and thus instead of looking at the entire distributions, it will be enough to look at the atoms when overlaps equal to 1. That is to say it will be enough in our case to consider the numbers

$$
\begin{aligned}
&\mathbb{E}\mu_{\beta,N}^{l+k_1+\dots+k_l}\Big(R_N(\sigma^1,\sigma^{l+1})=1,\dots,R_N(\sigma^1,\sigma^{l+k_1})=1,\dots,\\
&\qquad\dots,R_N(\sigma^l,\sigma^{l+k_1+\dots+k_{l-1}+1})=1,\dots,R_N(\sigma^l,\sigma^{l+k_1+\dots+k_l})=1\Big)\\
&=\mathbb{E}\mu_{\beta,N}^{l+k_1+\dots+k_l}\Big(\sigma^1=\sigma^{l+1},\dots,\sigma^1=\sigma^{l+k_1},\dots,\\
&\qquad\dots,\sigma^l=\sigma^{l+k_1+\dots+k_{l-1}+1},\dots,\sigma^l=\sigma^{l+k_1+\dots+k_l}\Big)\\
&=\mathbb{E}\mu_{\beta,N}^{l+k_1+\dots+k_l}\Big(\sigma^1=\sigma^{l+1}=\dots=\sigma^{l+k_1},\dots,\\
&\qquad\dots,\sigma^l=\sigma^{l+k_1+\dots+k_{l-1}+1}=\dots=\sigma^{l+k_1+\dots+k_l}\Big)
\end{aligned}
\tag{2.55}
$$

As we will show now, the multi-overlaps are not independent, but satisfy recursion relations that are due to rather general principles. It will be instructive to look at them in this simple context. These identities have been known in the physics literature and a more rigorous analysis is given in a paper by Ghirlanda and Guerra [GG]. Equivalent relations were in fact derived somewhat earlier by Aizenman and Contucci [AC]. See also [L]. The importance of these relations has been underlined by Talagrand [T4,T5]. Let us begin with the simplest instance of these relations.

***Proposition 2.8:*** *For any value of $\beta$,*

$$
\mathbb{E}\frac{d}{d\beta}F_{\beta,N}=-\beta(1-\mathbb{E}f_{\beta,N}(1))
\tag{2.56}
$$

***Proof:*** Obviously,

$$
\mathbb{E}\frac{d}{d\beta}F_{\beta,N}=-N^{-1}\mathbb{E}\frac{\mathbb{E}_\sigma\sqrt{N}X_\sigma e^{\beta\sqrt{N}X_\sigma}}{\mathbb{E}_\sigma e^{\beta\sqrt{N}X_\sigma}}
\tag{2.57}
$$

Now if $X$ is standard normal variable, and $g$ any function of at most polynomial growth, then

$$
\mathbb{E}[Xg(X)]=\mathbb{E}g'(X)
\tag{2.58}
$$

Using this identity in the right hand side of (2.57) with respect to the average over $X_\sigma$, we get immediately that

$$
\begin{aligned}
\mathbb{E}\frac{\mathbb{E}_\sigma\sqrt{N}X_\sigma e^{\beta\sqrt{N}X_\sigma}}{\mathbb{E}_\sigma e^{\beta\sqrt{N}X_\sigma}}&=N\beta\mathbb{E}\left(1-\frac{2^{-N}\mathbb{E}_\sigma e^{2\beta\sqrt{N}X_\sigma}}{(\mathbb{E}_\sigma e^{\beta\sqrt{N}X_\sigma})^2}\right)\\
&=N\beta\mathbb{E}\left(1-\mu_{\beta,N}^{\otimes 2}\left(\mathbb{1}_{\sigma^1=\sigma^2}\right)\right)
\end{aligned}
\tag{2.59}
$$

which is obviously the claim of the lemma.◇

In exactly the same way one can prove the following generalisation:

***Lemma 2.9:*** *Let $h : \mathcal{S}_N^n \to \mathbb{R}$ be any bounded function of $n$ spins. Then*

$$\begin{aligned}&\frac{1}{\sqrt{N}}\mathbb{E}\mu_{\beta,N}^{\otimes n}\left(X_{\sigma^k} h(\sigma^1,\dots,\sigma^n)\right)\\&=\beta\mathbb{E}\mu_{\beta,N}^{\otimes n+1}\left(h(\sigma^1,\dots,\sigma^n)\left(\sum_{l=1}^n \mathbb{1}_{\sigma^k=\sigma^l} - n\mathbb{1}_{\sigma^k=\sigma^{n+1}}\right)\right)\end{aligned}\tag{2.60}$$

***Proof:*** Left as an exercise.◇

The strength of Lemma 2.9 comes out when combined with a factorization result that in turn is a consequence of self-averaging.

***Lemma 2.10:*** *Let $h$ be as in the previous lemma. For all but possibly a countable number of values of $\beta$,*

$$\lim_{N\uparrow\infty}\frac{1}{\sqrt{N}}\left|\mathbb{E}\mu_{\beta,N}^{\otimes n}\left(X_{\sigma^k} h(\sigma^1,\dots,\sigma^n)\right) - \mathbb{E}\mu_{\beta,N}\left(X_{\sigma^k}\right)\mathbb{E}\mu_{\beta,N}^{\otimes n}\left(h(\sigma^1,\dots,\sigma^n)\right)\right| = 0 \tag{2.61}$$

***Proof:*** Let us write

$$\begin{aligned}&\left(\mathbb{E}\mu_{\beta,N}^{\otimes n}\left(X_{\sigma^k} h(\sigma^1,\dots,\sigma^n)\right) - \mathbb{E}\mu_{\beta,N}\left(X_{\sigma^k}\right)\mathbb{E}\mu_{\beta,N}^{\otimes n}\left(h(\sigma^1,\dots,\sigma^n)\right)\right)^2\\&=\left(\mathbb{E}\mu_{\beta,N}^{\otimes n}\left(\left(X_{\sigma^k}-\mathbb{E}\mu_{\beta,N}^{\otimes n}X_{\sigma^k}\right)h(\sigma^1,\dots,\sigma^n)\right)\right)^2\\&\leq \mathbb{E}\mu_{\beta,N}^{\otimes n}\left(X_{\sigma^k}-\mathbb{E}\mu_{\beta,N}^{\otimes n}X_{\sigma^k}\right)^2\mathbb{E}\mu_{\beta,N}^{\otimes n}\left(h(\sigma^1,\dots,\sigma^n)\right)^2\end{aligned}\tag{2.62}$$

where the last inequality is the Cauchy–Schwarz inequality applied to the joint expectation with respect to the Gibbs measure and the disorder. Obviously the first factor in the last line is equal to

$$\begin{aligned}&\mathbb{E}\left(\mu_{\beta,N}(X_\sigma^2)-[\mu_{\beta,N}(X_\sigma)]^2\right)+\mathbb{E}(\mu_{\beta,N}(X_\sigma)-\mathbb{E}\mu_{\beta,N}(X_\sigma))^2\\&=-\beta^{-2}\mathbb{E}\frac{d^2}{d\beta^2}F_{\beta,N}+N\beta^{-2}\mathbb{E}\left(\frac{d}{d\beta}F_{\beta,N}-\mathbb{E}\frac{d}{d\beta}F_{\beta,N}\right)^2\end{aligned}\tag{2.63}$$

We know that $F_{\beta,N}$ converges as $N\uparrow\infty$ and that the limit is infinitely differentiable for all $\beta \geq 0$, except at $\beta=\sqrt{2\ln 2}$; moreover, $-F_{\beta,N}$ is convex in $\beta$. Then standard results of convex analysis imply that

$$\limsup_{N\uparrow\infty}(-\mathbb{E}\frac{d^2}{d\beta^2}F_{\beta,N}) = -\frac{d^2}{d\beta^2}\lim_{N\uparrow\infty}\mathbb{E}F_{\beta,N} \tag{2.64}$$

which is finite for all $\beta \neq \sqrt{2\ln 2}$. Thus, the first term in (2.63) will vanish when divided by $N$. To see that the coefficient of $N$ of the second term gives a vanishing contribution, we use the general fact that if the variance of family of a convex (or concave) functions tends to zero, then the same is true for its derivative, except possibly on a countable set of values of their argument. In Theorem 2.2 we have more than established that the variance of $F_{\beta,N}$ tends to zero, and hence the result of the Lemma is proven. $\diamondsuit$

If we combine Proposition 2.8, Lemma 2.9, and Lemma 2.10 we arrive immediately at

***Proposition 2.11:****For all but a countable set of values $\beta$, for any bounded function $h$ : $\mathcal{S}_N^n \to \mathbb{R}$,*

$$\begin{aligned}\lim_{N\uparrow\infty}\Bigg|&\mathbb{E}\mu_{\beta,N}^{\otimes n+1}\left(h(\sigma^1,\dots,\sigma^n)\mathbb{1}_{\sigma^k=\sigma^{n+1}}\right)\\ &-\frac{1}{n}\mathbb{E}\mu_{\beta,N}^{\otimes n+1}\left(h(\sigma^1,\dots,\sigma^n)\left(\sum_{l\neq k}^{n}\mathbb{1}_{\sigma^l=\sigma^k}+\mathbb{E}\mu_{\beta,N}^{\otimes 2}(\mathbb{1}_{\sigma^1=\sigma^2})\right)\right)\Bigg|=0\end{aligned} \tag{2.65}$$

Together with the fact that the product Gibbs measures are concentrated only on the sets where the overlaps take values 0 and 1, (2.65) permits to compute the distribution of all higher overlaps in terms of the two-replica overlap. E.g., if we put

$$A_n \equiv \lim_{N\uparrow\infty}\mathbb{E}\mu_{\beta,N}^{\otimes n}(\mathbb{1}_{\sigma^1=\sigma^2=\dots=\sigma^n}) \tag{2.66}$$

then (2.65) with $h=\mathbb{1}_{\sigma^1=\sigma^2=\dots=\sigma^n}$ provides the recursion

$$\begin{aligned}A_{n+1}&=\frac{n-1}{n}A_n+\frac{1}{n}A_nA_2=A_n\left(1-\frac{1-A_2}{n}\right)\\ &=\prod_{k=2}^{n}\left(1-\frac{1-A_2}{k}\right)A_2\\ &=\frac{\Gamma(n+A_2)}{\Gamma(n+1)\Gamma(A_2)}\end{aligned} \tag{2.67}$$

Note that we can use alternatively Theorem 2.4 to compute, for the non-trivial case $\beta > \sqrt{2\ln 2}$,

$$\lim_{N\uparrow\infty}\mu_{\beta,N}^{\otimes 2}(\mathbb{1}_{\sigma^1=\sigma^2=\dots=\sigma^n})=\int\mathcal{K}_\beta(dm)[m(1)]^{n-1} \tag{2.68}$$

so that (2.67) implies a formula for the mean of the $n$-th moments of $\mathcal{W}$,

$$\mathbb{E}\int\mathcal{W}(dy,dw)w^n=\frac{\Gamma(n+A_2)}{\Gamma(n+1)\Gamma(A_2)} \tag{2.69}$$

where $A_2 = \mathbb{E}\int \mathcal{W}(dy,dw)w^2$. This result has been obtained by a direct computation by Ruelle ([Ru], Corollary 2.2), but its derivation via the Ghirlanda–Guerra identities shows a way to approach this problem in a different manner that has the potential to give results in more complicated situations.[5]

## 3. The Derrida models.

The reader of the previous chapter may think that that was 'much ado about nothing'. First, it was all about independent random variables, second, we used heavy tools to describe structure that is in fact very simple. We will now move towards a class of models that have been introduced 17 years ago by Derrida as "simplified" spin glass models. It turns out that while these models exhibit structure that is as complex as (and in fact almost identical to) in the Sherrington-Kirkpatrick type spin glasses, they can now be analysed with full rigor with the help of the tools I have explained in the previous section. The results of these Section cover recent work with Irina Kurkova [BK1,BK2,BK3]. The purpose of this section is to explain how the remarkable universal structures predicted by Parisi's replica symmetry breaking scheme arise as a limiting object in a spin glass model. For further analysis of the limiting object itself we refer to papers by Bolthausen and Sznitman [BoSz] and Bertoin and LeGall [BeLe].

### 3.1. Definitions and basics.

As we have already pointed out in the introduction, from a mathematical point of view it is natural to embed the SK models in the general setting of models based on Gaussian processes on the hypercubes $\mathcal{S}_N$. The special feature of the SK models in that context is then that their covariance depends only on the "overlap", $R_N(\sigma,\sigma') = \frac{1}{N}(\sigma,\sigma')$.

Derrida introduced another class of models that he called *Generalized Random Energy models* (GREM) that can be constructed in full analogy to the SK class by introducing another function characterizing distance that is to replace the overlap $R_N$, namely

$$\operatorname{dist}(\sigma,\sigma') \equiv \frac{1}{N}\left(\min(i : \sigma_i \neq \sigma_i') - 1\right) \tag{3.1}$$

To be precise, dist is an *ultrametric* valuation on the set $\mathcal{S}_N$. An ultrametric distance would be given e.g. by a function $D(\sigma,\sigma') = \exp(-\operatorname{dist}(\sigma,\sigma'))$. We will now consider centered

[5] More generally, one may derive recursion formulas for more general moments of Ruelle's process that show that the identities (2.65) determine completely the process of Ruelle in terms of the two-overlap distribution function.

Gaussian processes $X_\sigma$ on $\mathcal{S}_N$ those covariance is given as

$$\operatorname{cov}(X_\sigma, X_{\sigma'}) = \mathbb{E}X_\sigma X_{\sigma'} = A(\operatorname{dist}(\sigma,\sigma')) \tag{3.2}$$

where $A$ is a probability distribution function on the interval $[0,1]$.

In fact, the original models of Derrida correspond to the special case when $A$ is the distribution function of a random variable that takes only finitely many values, i.e. when $A$ is a monotone increasing step function with finitely many steps. However, Derrida also considered limits when the number of these steps tend to infinity.

The choice of the distance dist has a number of remarkable effect that help to make these models truly solvable. In particular, it allows to introduce a continuous time martingale $X_\sigma(t)$ those marginal at $t=1$ coincides with $X_\sigma$. This process is simply a Gaussian process on $\mathcal{S}_N \times [0,1]$ with covariance

$$\operatorname{cov}(X_\sigma(t), X_{\sigma'}(t')) = t \wedge t' \wedge A(\operatorname{dist}(\sigma,\sigma')) \tag{3.3}$$

In particular, this gives rise to the integral representation of $X_\sigma$ as

$$X_\sigma = \int_0^1 dX_\sigma(t) \tag{3.4}$$

where the increments satisfy

$$\mathbb{E}dX_\sigma(t)dX_{\sigma'}(t') = dtdt'\delta(t-t')\mathbb{1}_{A(\operatorname{dist}(\sigma,\sigma'))>t} \tag{3.5}$$

If $A$ is a step function, this gives rise to a representation in the form

$$X_\sigma \equiv \sqrt{a_1}X_{\sigma_1} + \sqrt{a_2}X_{\sigma_1\sigma_2} + \cdots + \sqrt{a_n}X_{\sigma_1\sigma_2\ldots\sigma_n}, \quad \text{if } \sigma = \sigma_1\sigma_2\ldots\sigma_n, \tag{3.6}$$

where $a_i$ is the increment of $A$ at the step point $q_i = \sum_{j=1}^i \frac{\ln \alpha_j}{\ln 2}$, and $\sigma = \sigma_1\sigma_2\ldots\sigma_n$ with $\sigma_i \in \{-1,1\}^{\ln \alpha_i N}$.

Note that in the SK class, neither is it possible to construct such a representation, nor are step functions allowed as covariances.

The representation (3.6) allows explicit computations of the partition function. This was done first by Derrida and Gardner [DG1], and in full generality (and with full rigor) by Cappocaccia, Cassandro, and Picco [CaCaPi]. While we will not reproduce this calculations (they are in spirit not very different from those in the REM and make use of (3.6) to set up a recursive scheme), we will state their result in a particularly useful form.

Let us denote the convex hull of the function $A(x)$ by $\bar{A}(x)$. We will also need the left-derivative of this function, $\bar{a}(x) \equiv \lim_{\epsilon\downarrow 0} \epsilon^{-1}(\bar{A}(x) - \bar{A}(x-\epsilon))$ which exists for all values of $x \in (0,1]$.

***Theorem 3.1:*** *Whenever $A$ is a step function with finitely many steps, the free energy $F_{\beta,N} \equiv \frac{1}{N}\ln Z_{\beta,N}$ converges almost surely to the non-random limit $F_\beta$ given by*

$$F_\beta = \sqrt{2\ln 2}\beta \int_0^{x_\beta} \sqrt{\bar{a}(x)}dx + \frac{\beta^2}{2}(1 - \bar{A}(x(\beta))) \tag{3.7}$$

*where*

$$x_\beta \equiv \sup\left(x|\bar{a}(x) > \frac{2\ln 2}{\beta^2}\right) \tag{3.8}$$

It is also very easy to derive from (3.7) an explicit formula for the distance-distribution function

$$f_{\beta,N}(x) \equiv \mu_{\beta,N}^{\otimes 2}(\,\mathrm{dist}(\sigma,\sigma') < x) \tag{3.9}$$

This just makes use of the fact that

***Proposition 3.2:*** *For any value of $\beta$, and any $i = 1,\dots,n$,*

$$\mathbb{E}\frac{d}{d\sqrt{a}_i}F_{\beta,N} = -\beta^2\sqrt{a_i}\mathbb{E}f_{\beta,N}(q < q_i) \tag{3.10}$$

*with the convention that $q_0 = 0$ and $q_n = 1$.*

This implies in fact immediately that

***Theorem 3.3:*** *Whenever $A$ is a step function with finitely many steps, the $f_{\beta,N}$ converges in mean to the limiting function $f_\beta$ with*

$$\mathbb{E}f_\beta(x) = \begin{cases} \beta^{-1}\sqrt{2\ln 2}/\sqrt{\bar{a}(x)}, & \text{if}\, x \le x_\beta \\ 1, & if\, x > x_\beta \end{cases} \tag{3.11}$$

It is obvious that if $A_n$ is a sequence of step functions that converges to a limiting function $A$, then the sequences of free energies and distance distributions converge. It in not very difficult to show [BK3] that these limits then are in fact the free energies and distribution functions for the corresponding models with arbitrary $A$. The results obtained here coincide with those of Derrida and Gardner, and in particular reproduce exactly the findings of the replica method [DG2].

**3.2. Gibbs measures and point processes.**

As in the case of the REM, Ruelle [Ru] had proposed an effective model for the thermodynamic limit of the GREM in terms of Poisson processes, or rather *"Poisson cascades"*, i.e. nested sequences of Poisson processes, without establishing a rigorous relation between the two models. Ruelle also constructed limiting objects of his processes when the number of "levels" (i.e. $n$) tends to infinity. The connection between Ruelle's models and the GREMs with finitely many levels have been made rigorous in [BK1]. While again in spirit the proofs are similar to those in the REM, they require considerably more computations.

However, it is quite remarkable that via the Ghirlanda-Guerra relations, one can construct (at least in principle) the thermodynamic limit on the level of the measures on the mass distribution without much explicit computation even in the case of continuous $A$. To prove these inequalities, we have to impose a "non-criticality" conditions on $A$: For any $x$ where the convex hull of $A$ is not strictly convex (i.e. where $\bar{A}$ is linear in neighborhood of $x$, $A(x) < \bar{A}(x)$). We assume this condition to hold in the remainder of the article.

It will be convenient to introduce here the analogues of the random measures $\mathcal{K}$ defined above where the overlap $R_N$ is replaced by the distance dist. I.e. we set now

$$m_\sigma(x) \equiv \mu_{\beta,N}(\sigma : \operatorname{dist}(\sigma', \sigma) > x) \tag{3.12}$$

and

$$\mathcal{K}_{\beta,N} \equiv \sum_{\sigma \in \mathcal{S}_N} \mu_{\beta,N}(\sigma) \delta_{m_\sigma(\cdot)} \tag{3.13}$$

In the case when $A$ is a step function with finitely many steps, one can control the convergence of $\mathcal{K}_{\beta,N}$ to a limit rather explicitly. We will present the corresponding results, without proof, below.

In the general case, this will no longer be possible. However, the Ghirlanda-Guerra identities will allow again to prove the existence of the limit and to describe its properties. The key point to notice is that to prove convergence, it is enough to prove convergence of all expressions of the form

$$\mathbb{E}\Bigg(\left(\int \mathcal{K}_{\beta,N}(dm) m(\Delta_{11})^{r_{11}} \dots m(\Delta_{1j_1})^{r_{1j_1}}\right)^{q_1} \dots \\ \dots \left(\int \mathcal{K}_{\beta,N}(dm) m(\Delta_{l1})^{r_{l1}} \dots m(\Delta_{lj_l})^{r_{lj_l}}\right)^{q_l}\Bigg) \tag{3.14}$$

where $\Delta_{ij} \subset [0,1]$ and $q_i, r_{ij}$ are integers.

The key is thus to establish again the Ghirlanda-Guerra identities. In this the process $X_\sigma(t)$ plays a crucial rôle. It will be convenient to use the time-changed process

$$Y_\sigma(t) \equiv X_\sigma(A(t)) \tag{3.15}$$

***Theorem 3.4:*** *For any $n \in \mathbb{N}$ and any $x \in [0,1] \backslash x_\beta$,*

$$\lim_{N\uparrow\infty} \Bigg| \mathbb{E}\mu_{\beta,N}^{\otimes n+1}\left(h(\sigma^1,\dots,\sigma^n) 1\!\!1_{A(\operatorname{dist}(\sigma^k,\sigma^{n+1}))\geq x}\right) \\ - \frac{1}{n}\mathbb{E}\mu_{\beta,N}^{\otimes n+1}\left(h(\sigma^1,\dots,\sigma^n)\left(\sum_{l\neq k}^n 1\!\!1_{A(\operatorname{dist}(\sigma^k,\sigma^l))\geq x} + \mathbb{E}\mu_{\beta,N}^{\otimes 2}(1\!\!1_{A(\operatorname{dist}(\sigma^1,\sigma^2))\geq x})\right)\right)\Bigg| = 0 \tag{3.16}$$

***Proof:*** As a first step we need the following lemma.

***Lemma 3.5:*** *Let $h : \mathcal{S}_N^n \to \mathbb{R}$ be any bounded function of $n$ spins. For any $t \in (0,1]$*

$$\frac{1}{\sqrt{N}}\mathbb{E}\mu_{\beta,N}^{\otimes n}\left(dY_{\sigma^k}(t) h(\sigma^1,\dots,\sigma^n)\right) \\ = \beta\mathbb{E}\mu_{\beta,N}^{\otimes n+1}\left(h(\sigma^1,\dots,\sigma^n)\left(\sum_{l=1}^n 1\!\!1_{\operatorname{dist}(\sigma^k,\sigma^l)\geq t} - n 1\!\!1_{\operatorname{dist}(\sigma^k,\sigma^{n+1})\geq t}\right)\right) dA(t) \tag{3.17}$$

***Proof:*** The proof makes use of the Gaussian integration by parts formula

$$\mathbb{E} dX_\sigma(t) f\left(\int dX_{\sigma'}(s)\right) = \mathbb{E} f'\left(\int dX_{\sigma'}(s)\right) \int \mathbb{E} dX_\sigma(t) dX_{\sigma'}(s) \\ = \mathbb{E} f'\left(X_{\sigma'}\right) 1\!\!1_{A(\operatorname{dist}(\sigma,\sigma'))\geq t} dt \tag{3.18}$$

where $f$ is any differentiable function. Note that the left hand side of (3.17) can be written as

$$N^{-1/2}\mathbb{E}\mathbb{E}_{\sigma^1\dots\sigma^n} h(\sigma^1,\dots,\sigma^n) dY_{\sigma^k}(t) \prod_{l=1}^n f\left(X_{\sigma^l}\right) \tag{3.19}$$

with

$$f\left(X_{\sigma^l}\right) = \frac{e^{\beta\sqrt{N}X_{\sigma^l}(1)}}{\mathbb{E}_{\sigma^l}\, e^{\beta\sqrt{N}X_{\sigma^l}(1)}} \tag{3.20}$$

Using (3.18) gives readily

$$\begin{aligned}&\frac{1}{\sqrt{N}}\mathbb{E}\mu_{\beta,N}^{\otimes n}\left(dY_{\sigma^k}(t)h(\sigma^1,\dots,\sigma^n)\right)\\&=\beta\mathbb{E}\mu_{\beta,N}^{\otimes n+1}\left(h(\sigma^1,\dots,\sigma^n)\left(\sum_{l=1}^n \mathbb{1}_{A(\operatorname{dist}(\sigma^k,\sigma^l))\geq t}-n\mathbb{1}_{A(\operatorname{dist}(\sigma^k,\sigma^{n+1}))\geq t}\right)\right)dt\end{aligned} \tag{3.21}$$

Realizing that $A(\operatorname{dist}(\sigma,\sigma')) < A(t)$ is equivalent to $\operatorname{dist}(\sigma,\sigma') < t$ whenever $A(t)$ is not constant then yields the claim of the lemma. $\diamondsuit$

The more important step of the proof is contained in the next lemma.

***Lemma 3.6:*** *Let $h$ be as in the previous lemma. Except possibly when $t = x_\beta$,*

$$\begin{aligned}\lim_{N\uparrow\infty}\frac{1}{\sqrt{N}}\Big|&\mathbb{E}\mu_{\beta,N}^{\otimes n}\left((Y_{\sigma^k}(t)-Y_{\sigma^k}(t-\epsilon))h(\sigma^1,\dots,\sigma^n)\right)\\&-\mathbb{E}\mu_{\beta,N}\left(Y_{\sigma^k}(t)-Y_{\sigma^k}(t-\epsilon)\right)\mathbb{E}\mu_{\beta,N}^{\otimes n}\left(h(\sigma^1,\dots,\sigma^n)\right)\Big|=0\end{aligned} \tag{3.22}$$

***Proof:*** Let us write

$$\begin{aligned}&\left(\mathbb{E}\mu_{\beta,N}^{\otimes n}\left(Y_{\sigma^k}(t)-Y_{\sigma^k}(t-\epsilon)\right)-\mathbb{E}\mu_{\beta,N}\left(Y_{\sigma^k}(t)-Y_{\sigma^k}(t-\epsilon)\right)\mathbb{E}\mu_{\beta,N}^{\otimes n}\left(h(\sigma^1,\dots,\sigma^n)\right)\right)^2\\&=\left(\mathbb{E}\mu_{\beta,N}^{\otimes n}\left(\left((Y_{\sigma^k}(t)-Y_{\sigma^k}(t-\epsilon))-\mathbb{E}\mu_{\beta,N}^{\otimes n}(Y_{\sigma^k}(t)-Y_{\sigma^k}(t-\epsilon))\right)h(\sigma^1,\dots,\sigma^n)\right)\right)^2\\&\leq\mathbb{E}\mu_{\beta,N}^{\otimes n}\left((Y_{\sigma^k}(t)-Y_{\sigma^k}(t-\epsilon))-\mathbb{E}\mu_{\beta,N}^{\otimes n}(Y_{\sigma^k}(t)-Y_{\sigma^k}(t-\epsilon))\right)^2\mathbb{E}\mu_{\beta,N}^{\otimes n}\left(h(\sigma^1,\dots,\sigma^n)\right)^2\end{aligned} \tag{3.23}$$

where the last inequality is the Cauchy–Schwarz inequality applied to the joint expectation with respect to the Gibbs measure and the disorder. Obviously the first factor in the last line is equal to

$$\begin{aligned}&\mathbb{E}\mu_{\beta,N}\left((Y_{\sigma^k}(t)-Y_{\sigma^k}(t-\epsilon))-\mu_{\beta,N}(Y_{\sigma^k}(t)-Y_{\sigma^k}(t-\epsilon))\right)^2\\&\quad+\mathbb{E}\left(\mu_{\beta,N}(Y_{\sigma^k}(t)-Y_{\sigma^k}(t-\epsilon))-\mathbb{E}\mu_{\beta,N}(Y_{\sigma^k}(t)-Y_{\sigma^k}(t-\epsilon))\right)^2\end{aligned} \tag{3.24}$$

Now let us introduce the deformed process

$$X_\sigma^u \equiv X_\sigma + u\left(Y_\sigma(t)-Y_\sigma(t-\epsilon)\right) \tag{3.25}$$

If we denote by $F^u_{\beta,N}$ the free energy corresponding to this deformed process, the last line of (3.24) can be represented as

$$\beta^{-2}\mathbb{E}\frac{d^2}{du^2}F^u_{\beta,N}+N\beta^{-2}\mathbb{E}\left(\frac{d}{du}F^u_{\beta,N}-\mathbb{E}\frac{d}{du}F^u_{\beta,N}\right)^2 \tag{3.26}$$

At this point we need a concentration result on the free energy which we state here without proof.

***Lemma 3.7:*** *For any $\beta$, and any covariance distribution $A$, for any $r\geq 0$*

$$\mathbb{P}\left[|F_{\beta,N}-\mathbb{E}F_{\beta,N}|>r\right]\leq 2\exp\left(-\frac{r^2N}{2\beta^2}\right) \tag{3.27}$$

$F^u_{\beta,N}$ converges as $N\uparrow\infty$ and that the limit is infinitely differentiable as a function of $u$, except possibly when $x_\beta=t$, provided $A$ satisfies the non-criticality condition; moreover, $-F^u_{\beta,N}$ is convex in the variable $u$. This can be seen by explicit computation using the expression (3.7) for the free energy. Then a standard result of convex analysis (see [Ro], Theorem 25.7) imply that

$$\limsup_{N\uparrow\infty}(-\mathbb{E}\frac{d^2}{du^2}F^u_{\beta,N})=-\frac{d^2}{du^2}\lim_{N\uparrow\infty}\mathbb{E}F^u_{\beta,N} \tag{3.28}$$

which is finite at zero except possibly if $x_\beta=t$. Thus, the first term in (2.63) will vanish when divided by $N$. To see that the coefficient of $N$ of the second term gives a vanishing contribution, we use the general fact that if the variance of family of a convex (or concave) functions tends to zero, then the same is true for its derivative, provided the second derivative of the expectation is bounded (see e.g. Lemma 8.9 in [BG], or Proposition 4.3 in [T2]).

But by Lemma 3.7 the variance of $F_{\beta,N}$ tends to zero, and (3.28) implies that $\mathbb{E}\frac{d^2}{du^2}F^u_{\beta,N}$ is bounded for large enough $N$ whenever $\frac{d^2}{du^2}\mathbb{E}F^u_\beta$ is finite. Hence the result of the lemma is proven. $\diamondsuit$

To prove the theorem we use integrate (3.17) and then use (3.22) on the left hand side. This gives

$$\begin{aligned}\lim_{N\uparrow\infty}\Bigg(&\frac{1}{\sqrt{N}}\mathbb{E}\mu^{\otimes n}_{\beta,N}\left(Y_{\sigma^k}(t)-Y_{s^k}(t-\epsilon)\right)\mathbb{E}\mu^{\otimes n}_{\beta,N}\left(h(\sigma^1,\dots,\sigma^n)\right)\\&-\beta\int_{t-\epsilon}^t\left(\mathbb{E}\mu^{\otimes n+1}_{\beta,N}\left(h(\sigma^1,\dots,\sigma^n)\left(\sum_{l=1}^n\mathbb{1}_{\mathrm{dist}(\sigma^k,\sigma^l)\geq s}-n\mathbb{1}_{\mathrm{dist}(\sigma^k,\sigma^{n+1})\geq s}\right)\right)\right)dA(s)\Bigg)=0\end{aligned} \tag{3.29}$$

Finally, we use once more (3.17) with $n=1$ to express $\mathbb{E}\mu_{\beta,N}^{\otimes n}\left(Y_{\sigma^k}(t)-Y_{s^k}(t-\epsilon)\right)$ in terms of the two replica distribution. The final result follows by trivial algebraic manipulations and the fact that $\epsilon$ is arbitrary. $\diamondsuit\diamondsuit$

Following [GG], we now define the family of measures $\mathbb{Q}_N^{(n)}$ on the space $[0,1]^{n(n-1)/2}$.

$$\mathbb{Q}_{\beta,N}^{(n)}\left(\underline{\text{dist}}\in\mathcal{A}\right)\equiv\mathbb{E}\mu_{N,\beta}^{\otimes n}\left[\underline{\text{dist}}\in\mathcal{A}\right] \tag{3.30}$$

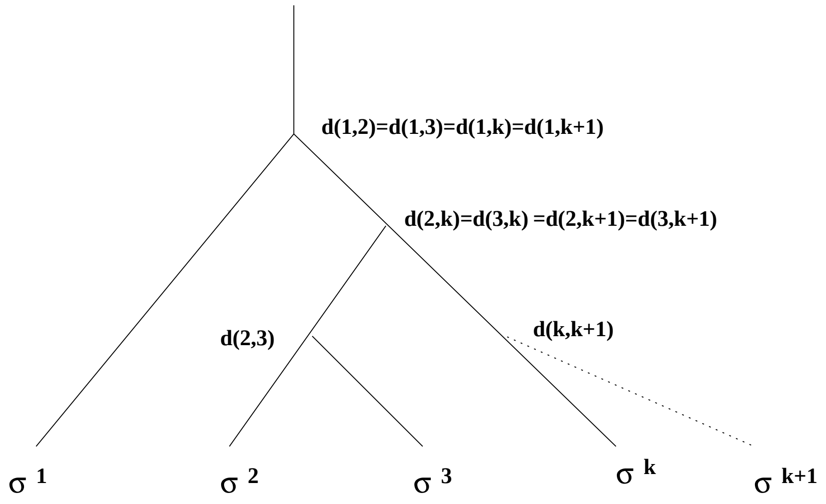
where $\underline{\text{dist}}$ denotes the vector of replica distances whose components are $\text{dist}(\sigma^l,\sigma^k)$, $1\leq l<k\leq n$. Denote by $\mathcal{B}_k$ the sigma-algebra generated by the first $k(k-1)/2$ coordinates, and let $A$ be a Borel set in $[0,1]$.

***Theorem 3.8:*** *The family of measures $\mathbb{Q}_{\beta,N}^{(n)}$ converge to limiting measures $\mathbb{Q}_\beta^{(n)}$ for all finite $n$, as $N\uparrow\infty$. Moreover, these measures are uniquely determined by the distance distribution functions $f_\beta$. They satisfy the identities*

$$\mathbb{Q}_\beta^{(n+1)}\left(d_{k,n+1}\in A|\mathcal{B}_n\right)=\frac{1}{n}\mathbb{Q}_\beta^{(2)}(A)+\frac{1}{n}\sum_{l\neq k}^{n}\mathbb{Q}_\beta^{(n)}\left(d_{k,l}\in A|\mathcal{B}_n\right) \tag{3.31}$$

*for any Borel set $A$.*

***Proof:*** Choosing $h$ as the indicator function of any desired event in $\mathcal{B}_k$, one sees that (3.16) implies (3.31). This actually implies that in the limit $N\uparrow\infty$, the family of measures $\mathbb{Q}_{\beta,N}^{(n)}$ is entirely determined by the two-replica distribution function. While this may not appear obvious, it follows when taking into account the ultrametric property of the function dist. This is most easily seen by realising that the prescription of the mutual distances between $k$ spin configurations amounts to prescribing a tree (start all $k$ configurations at the origin and continue on top of each other as long as the coordinates coincide, then branch of). To determine the full tree of $k+1$ configurations, it is sufficient to know the overlap of configuration $\sigma^{(k+1)}$ with the configuration it has maximal overlap with, since then all overlaps with all other configurations are determined. But the corresponding probabilities can be computed recursively via (3.31).




**The distance d(k,k+1) determines all other distances d(j,k+1)**

Now we have already seen that $\mathbb{Q}^{(2)}_{\beta,N} = \mathbb{E}f_{\beta,N}$ converges. Therefore the relation (3.31) implies the convergence of all distributions $\mathbb{Q}^{(n)}_{\beta,N}$, and proves the relation (3.31) hold for the limiting measures. $\diamondsuit$

Now it is clear that all expressions of the form (2.38) (with $R_N$ replaced by dist) can be expressed in terms of the measures $\mathbb{Q}^{(k)}_{\beta,N}$ for $k$ sufficiently large (we leave this as an exercise for the reader to write down). Thus, Theorem 3.8 implies in turn the convergence of the process $\mathcal{K}_{\beta,N}$ to a limit $\mathcal{K}_\beta$.

A remarkable feature takes place again if we are only interested in the marginal process $K_\beta(t)$ for fixed $t$. This process is a simple point process on $[0,1]$ and is fully determined in terms of the moments

$$
\begin{aligned}
&\mathbb{E}\left(\int K_{\beta,N}(t)(dx)x^{r_1}\cdots\int K_{\beta,N}(t)(dx)x^{r_j}\right)\\
&=\mathbb{E}\mu_{\beta,N}^{\otimes r_1+\cdots+r_j+j}\Big(\operatorname{dist}(\sigma^1,\sigma^{j+1})>t,\dots,\operatorname{dist}(\sigma^1,\sigma^{j+r_1})>t,\dots,\\
&\qquad \dots,\operatorname{dist}(\sigma^j,\sigma^{j+r_1+\cdots+r_{j-1}+1})>t,\dots\operatorname{dist}(\sigma^j,\sigma^{j+r_1+\cdots+r_j})>t\Big)
\end{aligned}
\tag{3.32}
$$

This restricted family of moments satisfies via the Ghirlanda-Guerra identities exactly the same recursion as in the case of the REM. This implies:

***Theorem 3.9:*** *Assume that $t$ is such that $\mathbb{E}\mu_\beta^{\otimes 2}(\operatorname{dist}(\sigma,\sigma')<t)=1/\alpha>0$. Then the random measure $K_\beta(t)$ is a Dirichlet-Poisson process (see e.g.* [Ru,T1]*) with parameter $\alpha$ .*

In fact much more is true. We can consider the processes on arbitrary finite dimensional marginals, i.e.

$$K_{\beta,N}(t_1,\dots,t_m) \equiv \sum_{\sigma\in\mathcal{S}_N} \mu_{\beta,N}(\sigma)\delta_{m_\sigma(t_1),\dots,m_\sigma(t_m)} \tag{3.33}$$

for $0 < t_1 < \dots < t_m < 1$. The point is that this process is entirely determined by the expressions (3.14) with the $\Delta_{ij}$ all of the form $(t_i, 1]$ for $t_i$ in the fixed set of values $t_1,\dots,t_m$. This in turn implies that the process is determined by the multi-replica distribution functions $\mathbb{Q}^{(n)}_{\beta,N}$ restricted to the discrete set of events $\{\operatorname{dist}(\sigma^i,\sigma^j) > t_k\}$. Since these numbers are totally determined through the Ghirlanda-Guerra identities, they are identically to those obtained in a GREM with $m$ levels, i.e. a function $A$ having steps at the values $t_i$, those two-replica distribution function takes the same values as that of the model with continuous $A$ at the points $t_i$ and is constant between those values. In fact

***Theorem 3.10:*** *Let $0 < t_1 < \dots < t_k \leq q_{max}(\beta)$ be points of increase of $\mathbb{E}f_\beta$. Consider a GREM with $k$ levels and parameters $\alpha_i, a_i$ and temperature $\tilde\beta$ that satisfy $\ln\alpha_i/\ln 2 = t_i - t_{i-1}$, $\tilde\beta^{-1}\sqrt{2\ln\alpha_i/a_i} = \mathbb{E}f_\beta(t_i)$. Then*

$$\lim_{N\uparrow\infty} \mathcal{K}_{\beta,N}(t_1,\dots,t_k) = \mathcal{K}^{(k)}_{\tilde\beta} \tag{3.34}$$

Thus, if the $t_i$ are chosen in such a way that for all of them $\mathbb{E}f_\beta(t_i) > 0$, then we can construct an explicit representation of the limiting marginal process $\mathcal{K}_\beta(t_1,\dots,t_m)$ in terms of a Poisson-cascade process via the corresponding formulae in the associated $m$-level GREM. This construction is done in the next section. In this sense we obtain an explicit description of the limiting mass distribution function $\mathcal{K}_\beta$.

### 3.2. Probability cascades in the GREM with finitely many levels.

Let us now briefly explain the structure of the process $\mathcal{K}_\beta$ in the case when $A_n$ is a step function with steps of hight $a_i$ at the values $t_i \equiv \frac{\ln\alpha_i}{\ln 2}$. To avoid complications, we will assume that the linear interpolation of this function is convex, and that all points $t_i$ belong to the extremal set of the convex hull.

**Remark:** We will not give the proofs here, that are somewhat involved, in particular when the general case is considered. They can be found in [BK1, BK2]. The following summary of results is in fact just a cooked down version of the complete analysis of the GREM with finitely many hierarchies given there. Note that we draw heavily on the representation (3.6).

We introduce the function $u_{\ln\alpha,N}(x)$, $x\in\mathbb{R}$ as

$$u_{\ln\alpha,N}(x)=\sqrt{2\ln\alpha N}+\frac{x}{\sqrt{2\ln\alpha N}}-\frac{\ln N+\ln\ln\alpha+\ln 4\pi}{2\sqrt{2\ln\alpha N}}. \tag{3.35}$$

Note that then for all $i$,

$$\sum_{\sigma_i}\delta_{u^{-1}_{\ln\alpha_i,N}(X_{\sigma_1\cdots\sigma_{i-1}\sigma_i})}\to\mathcal{P}_i \tag{3.36}$$

where $\mathcal{P}_i$ are all independent Poisson point processes on $\mathbb{R}$ with intensity measure $e^{-x}dx$. Then under the assumptions on $A$, the following result holds:

***Theorem 3.11:*** *The following point processes on $\mathbb{R}^k$*

$$\mathcal{P}_N^{(k)}\equiv\sum_{\sigma_1}\delta_{u^{-1}_{\ln\alpha_1,N}(Y_{\sigma_1})}\sum_{\sigma_2}\delta_{u^{-1}_{\ln\alpha_2,N}(Y_{\sigma_1\sigma_2})}\cdots\sum_{\sigma_k}\delta_{u^{-1}_{\ln\alpha_k,N}(Y_{\sigma_1\sigma_2\cdots\sigma_k})}\to\mathcal{P}^{(k)}$$

*converge weakly to point process $\mathcal{P}^{(k)}$ on $\mathbb{R}^k$, which is characterised by the following generating functions:*

$$\begin{aligned}F_{\Delta_1\times\cdots\times\Delta_k}(z)&\equiv\mathbb{E}z^{\sum_{x_1}\mathbb{1}_{\{x_1\in\Delta_1\}}\cdots\sum_{x_k}\mathbb{1}_{\{x_k\in\Delta_k\}}}\\&=f_{1,\Delta_1}(f_{2,\Delta_2}(f_{3,\Delta_3}\cdots(f_{k-1,\Delta_{k-1}}(f_{k,\Delta_k}(z)))\cdots)),\qquad |z|<1\end{aligned} \tag{3.37}$$

*where $f_{i,\Delta_i}(z)=e^{K_i(z-1)(e^{-a_i}-e^{-b_i})}$, $\Delta_i=(a_i,b_i]$ with $a_i,b_i\in\mathbb{R}$ or $b_i=\infty$, $i=1,2,\ldots,k$.*

*Moreover, the following independence properties of the counting random variables of the process $\mathcal{P}^{(k)}$, $\sum_{x_1}\mathbb{1}_{\{x_1\in\Delta_1^j\}}\cdots\sum_{x_k}\mathbb{1}_{\{x_k\in\Delta_k^j\}}$, corresponding to the intervals $\Delta_1^j\times\cdots\times\Delta_k^j$, $\Delta_i^j=[a_i^j,b_i^j)$, $j=1,2,\ldots,k$, $k>1$, hold true:*

*(i) If the first components of these intervals are disjoint, i.e. $a_1^1\le b_1^1\le a_1^2\le b_1^2\le\cdots a_1^k\le b_1^k$, then these r.v. are independent.*

*(ii) If the first $l-1$ components of these intervals coincide and the $l$th components are disjoint, i.e. $\Delta_i^1=\cdots=\Delta_i^k$ for $i=1,\ldots,l-1$ and $a_l^1\le b_l^1\le a_l^2\le b_l^2\le\cdots a_l^k\le b_l^k$, then these r.v. are conditionally independent under condition that $\sum_{x_1}\mathbb{1}_{\{x_1\in\Delta_1\}}\cdots\sum_{x_{l-1}}\mathbb{1}_{\{x_{l-1}\in\Delta_{l-1}\}}$ is fixed.*

**Remark:** This theorem was proven for $k=2$ in [GMP].

We would like to clarify an intuitive construction of the process $\mathcal{P}$. If $k=1$, this is just a Poisson point process on $\mathbb{R}$ with intensity measure $K_1e^{-x}dx$. To construct $\mathcal{P}$ on $\mathbb{R}^2$ for $k=2$ we place the process $\mathcal{P}$ for $k=1$ on the axis of the first coordinate and through each of its points draw a straight line parallel to the axis of the second coordinate. Then we put on each of these lines independently a Poisson point process with intensity measure $K_2 e^{-x}dx$. These points on $\mathbb{R}^2$ form the process $\mathcal{P}$ with $k = 2$. Whenever $\mathcal{P}$ is constructed for $k-1$, we place it on the plane of the first $k-1$ coordinates and through each of its points draw a straight line parallel to the axis of the $n$th coordinate. On each of these lines we put after independently a Poisson point process with intensity measure $K_k e^{-x}dx$. These points constitute $\mathcal{P}$ on $\mathbb{R}^k$. Indeed, the projection of $\mathcal{P}^{(k)}$ in $\mathbb{R}^k$ to the plane of the first $\ell$ coordinates is distributed as the process $\mathcal{P}^{(\ell)}$ in $\mathbb{R}^\ell$.

We are now also in the position to formulate a result on the extreme order statistics of the random variables $X_\sigma$.

Let $\gamma_l \equiv \sqrt{a_l}/\sqrt{2\ln\alpha_l}$, $l = 1, 2, \dots, n$. By our assumption on $A$, $\gamma_1 > \gamma_2 > \dots > \gamma_n$. Define the function $U_{J,N}$ by

$$U_{J,N}(x) \equiv \sum_{l=1}^{n} \left( \sqrt{2Na_l \ln\alpha_l} - N^{-1/2}\gamma_l(\ln(N(\ln\alpha_l)) + \ln 4\pi)/2 \right) + N^{-1/2}x \tag{3.38}$$

and the point process

$$\mathcal{E}_N \equiv \sum_{\sigma\in\{-1,1\}^N} \delta_{U_{J,N}^{-1}(X_\sigma)}. \tag{3.39}$$

Then the following holds true:

***Theorem 3.12:*** *The point process $\mathcal{E}_N$ converges weakly, as $N \uparrow \infty$, to the point process on $\mathbb{R}$*

$$\mathcal{E} \equiv \int_{\mathbb{R}^n} \mathcal{P}^{(n)}(dx_1, \dots, dx_n)\delta_{\sum_{l=1}^{m}\gamma_l x_l} \tag{3.40}$$

*where $\mathcal{P}^{(n)}$ is the Poisson cascade introduced in Theorem 3.11.*

Next we state a convergence result for the partition function that is analogous to the low-temperature result Theorem 2.2, (v), in the REM.

One would be tempted to believe that the process that is relevant for the extremal process will again be the right one to choose. However, this will be the case only for large enough $\beta$. However, only the first $l(\beta)$ levels of the process participate, where

$$l(\beta) \equiv \max\{l \geq 1 : \beta^2\gamma_l > 1\} \tag{3.41}$$

and $l(\beta) \equiv 0$ if $\beta^2\gamma_l \leq 1$.

The following theorem yields the fluctuations of the partition function and connects the GREM to Ruelle's processes.

***Theorem 3.13:*** *With the definitions above, under our hypothesis on $A$,*

$$\begin{aligned} &e^{\sum_{j=1}^{l(\beta)}\left(-\beta N\sqrt{2a_j \ln \alpha_j}+\beta\gamma_j[\ln(N\ln\alpha_j)+\ln 4\pi]/2+N\ln\alpha_j\right)-N\sum_{i=l(\beta)+1}^{n}\beta^2 a_i/2} Z_{\beta,N} \\ &\stackrel{\mathcal{D}}{\to} C(\beta)\int_{\mathbb{R}^{l(\beta)}} e^{\beta\gamma_1 x_1+\beta\gamma_2 x_2+\cdots+\beta\gamma_{l(\beta)}x_{l(\beta)}}\mathcal{P}^{(l(\beta))}(dx_1\dots dx_{l(\beta)}). \end{aligned} \tag{3.42}$$

*This integral is over the process $\mathcal{P}^{(l(\beta))}$ on $\mathbb{R}^{l(\beta)}$ constructed in Theorem 3.11 . The constant $C(\beta)$ satisfies*

$$C(\beta)=1, \quad \textit{if} \quad \beta\gamma_{l(\beta)+1}<1, \tag{3.43}$$

*and*

$$C(\beta)=P\Big(\bigcap_{\substack{i:l(\beta)+1\le i\le l(\beta)+1 \\ (a_{l(\beta)+1}+\dots+a_i)/a_{l(\beta)+1}=\ln(\alpha_{l(\beta)+1}\cdots\alpha_i)/\ln\bar{\alpha}_{l(\beta)+1}}}(\sqrt{a_{l(\beta)+1}}Z_{l(\beta)+1}+\dots+\sqrt{a_i}Z_i<0)\Big) \tag{3.44}$$

*if $\beta\gamma_{l(\beta)+1}=1$*
*where $Z_{l(\beta)+1},\dots,Z_{l(\beta)+1}$ are independent standard Gaussian r.v. Moreover*

$$\ln Z_{N,\beta}-\mathbb{E}\ln Z_{N,\beta}\stackrel{\mathcal{D}}{\to}\ln C(\beta)\int_{\mathbb{R}^{l(\beta)}} e^{\beta\gamma_1 x_1+\beta\gamma_2 x_2+\cdots+\beta\gamma_{l(\beta)}x_{l(\beta)}}\mathcal{P}(dx_1\dots dx_{l(\beta)}).$$

Let us introduce the sets

$$B_l(\sigma)\equiv\{\sigma'\in\mathcal{S}_N:\ \mathrm{dist}(\sigma,\sigma')\ge q_l\} \tag{3.45}$$

We define point processes $\mathcal{W}^m_{\beta,N}$ on $(0,1]^m$ given by

$$\mathcal{W}^m_{\beta,N}\equiv\sum_\sigma \delta_{(\mu_{\beta,N}(B_1(\sigma)),\dots,\mu_{\beta,N}(B_m(\sigma)))}\frac{\mu_{\beta,N}(\sigma)}{\mu_{\beta,N}(B_m(\sigma))} \tag{3.46}$$

as well as their projection on the last coordinate,

$$\mathcal{R}^m_{\beta,N}\equiv\sum_\sigma \delta_{\mu_{\beta,N}(B_m(\sigma))}\frac{\mu_{\beta,N}(\sigma)}{\mu_{\beta,N}(B_m(\sigma))} \tag{3.47}$$

It is easy to see that the processes $\mathcal{W}^m_{\beta,N}$ satisfy

$$\mathcal{W}^m_{\beta,N}(dw_1,\dots,dw_m)=\int_0^1 W^{m+1}_{\beta,N}(dw_1,\dots,dw_m,dw_{m+1})\frac{w_{m+1}}{w_m} \tag{3.48}$$

where the integration is of course over the last coordinate $w_{m+1}$. Note that these processes will in general not all converge, but will do so only when for some $\sigma$, $\mu_\beta(B_m(\sigma))$ is strictly positive. From our experience with the partition function, it is clear that this will be the case precisely when $m \leq l(\beta)$. In fact, we will prove that

**Theorem 3.14:** *If $m \leq l(\beta)$, the point process $\mathcal{W}^m_{\beta,N}$ on $(0,1]^m$ converges weakly to the point process $\mathcal{W}^m_\beta$ whose atoms $w(i)$ are given in terms of the atoms $(x_1(i),\dots,x_m(i))$ of the point process $\mathcal{P}^{(m)}$ by*

$$\begin{aligned}&(w_1(i),\dots,w_m(i))\\ &=\left(\frac{\int \mathcal{P}^{(m)}(dy)\delta(y_1-x_1(i))e^{\beta(\gamma,y)}}{\int \mathcal{P}^{(m)}(dy)e^{\beta(\gamma,y)}},\dots,\frac{\int \mathcal{P}^{(m)}(dy)\delta(y_1-x_1(i))\dots\delta(y_m-x_m(i))e^{\beta(\gamma,y)}}{\int \mathcal{P}^{(m)}(dy)e^{\beta(\gamma,y)}}\right)\end{aligned} \tag{3.49}$$

*and the point processes $\mathcal{R}^{(m)}_{\beta,N}$ converge to the point process $\mathcal{R}^{(m)}_\beta$ whose atoms are the last component of the atoms in (3.49).*

Of course the most complete object we can reasonably study is the process $\widehat{\mathcal{W}}_\beta \equiv \mathcal{W}^{l(\beta)}_\beta$. Analogously, we will set $\widehat{\mathcal{R}}_\beta \equiv \mathcal{R}^{l(\beta)}_\beta$.

The point processes $\widehat{\mathcal{W}}^{(m)}_\beta$ takes values on vectors whose components form increasing sequences in $(0,1]$. Moreover, these atoms are naturally clustered in a hierarchical way. These processes were introduced by Ruelle [Ru] and called *probability cascades.* Finally, our last theorem gives the explicit construction of the limiting process $\mathcal{K}_\beta$ in the case of the step-function $A$ via Ruelle's probability cascades.

**Theorem 3.15:** *The process $\mathcal{K}_{\beta,N}$ converges to the process $\mathcal{K}_\beta$ which is supported on measures $\delta_{m(w)}$ indexed by points $w=(w(1),\dots,w(l(\beta)))\in\widehat{\mathcal{W}}_\beta$. More precisely*

$$\mathcal{K}_\beta = \int_{\mathbb{R}^{l(\beta)}} \widehat{\mathcal{W}}_\beta(dw)w(l(\beta))\delta_{m(w)}.$$

*where the measure $m(w)$ is given by the formula*

$$m(w) = (1-w_1)\delta_0 + (w_1-w_2)\delta_{\ln\alpha_1/\ln 2} + \dots + w_{l(\beta)}\delta_{\ln(\alpha_1\cdots\alpha_{l(\beta)})/\ln 2}$$

## References

[AC] M. Aizenman, P. Contucci. On the stability of the quenched state in mean field spin glass models. J. Stat. Phys. 92, 765-783 (1998).

[ALR] M. Aizenman, J.L. Lebowitz, D. Ruelle Some rigorous results on Sherrington-Kirkpatrick spin glass model. Commun. Math. Phys. 112 (1987), 3-20.

[BeLe] J. Bertoin, J.-F. Le Gall. The Bolthausen-Sznitman coalescent and the genealogy of continuous-state branching processes. Probab. Theory Related Fields 117, 249-266 (2000).

[BoSz] E. Bolthausen and A.-S. Sznitman. On Ruelle's probability cascades and an abstract cavity method. Comm. Math. Phys. 197 (1998), 247–276.

[B] A. Bovier. Statistical mechanics of disordered systems. MaPhySto Lecture Notes 10. Aarhus, 2002.

[BG] A. Bovier, V. Gayrard. The Hopfield model as a generalized random mean field model. In Mathematics of spin glasses and neural networks, A. Bovier, P. Picco. Eds., Progress in Probablity, Birkhäuser, Boston, (1997).

[BKL] A. Bovier, I. Kurkova, M. Löwe. The fluctuations of the free energy in the REM and the $p$-spin SK models, Ann. Probab. **30**, 605-651 (2002).

[BK1] A. Bovier and I. Kurkova. Derrida's Generalized Random Energy models 1: Poisson cascades and extremal processes. preprint Université Paris 6 (2002).

[BK2] A. Bovier and I. Kurkova. Derrida's Generalized Random Energy models 2: Gibbs measures and probability cascades. preprint Université Paris 6 (2002).

[BK3] A. Bovier and I. Kurkova. Derrida's Generalized Random Energy models 3: Models with continuous hierarchies. preprint Université Paris 6 (2002).

[CN] F. Comets, J. Neveu. The Sherrington-Kirkpatrick Model of Spin Glasses and Stochastic Calculus: The High Temperature Case. Commun Math. Phys. 166 (1995), 549-564.

[D1] B. Derrida. Random energy model: limit of a family of disordered models, Phys. Rev. Letts. 45(1980), 79-82.

[D2] B. Derrida. Random energy model: An exactly solvable model of disordered systems, Phys. Rev. B 24, 2613-2626 (1981).

[D3] B. Derrida, A generalization of the random energy model that includes correlations between the energies, J. Physique. Lett. **46**, 401–407 (1985).

[DG1] B. Derrida and E. Gardner, Solution of the generalized random energy model, J. Phys. C **19**, 2253–2274 (1986).

[DG2] B. Derrida and E. Gardner, Magnetic properties and function $q(x)$ of the generalised random energy model, J. Phys. C **19**, 5783-5798 (1986).

[DG3] E. Gardner and B. Derrida, The probability distribution of the partition function of the random energy model, J. Phys. A**22**, 1975-1981 (1989)

[DW] T.C. Dorlas, J.R. Wedagedera. Large deviations and the random energy model. Int. J. Mod. Phys. B 15 (2001), 1-15.

[Ei] Th. Eisele. On a third-order phase transition. Comm. Math. Phys. 90 (1983), 125-159.

[GMP] A. Galvez, S. Martinez, P. Picco. Fluctuations in Derrida's random energy and generalized random energy models, J. Stat. Phys. 54 (1989), 515-529.

[GG] S. Ghirlanda, F. Guerra. General properties of the overlap probability distributions in disordered spin systems.Towards Parisi ultrametricity, J. Phys. A 31(1998) , 9144-9155.

[GM] D.J. Gross, M. Mézard. The simplest spin glass. Nucl. Phys. B 240 (1984), 431-452.

[G] F. Guerra. Broken Replica Symmetry Bounds in the Mean Field Spin Glass Model. preprint `cond-mat/0205123` (2002).

[GT] F. Guerra, F.L. Toninelli. The thermodynamic limit in mean field spin glass models. preprint `cond-mat/0204280`.

[K] J.F.C. Kingman. Poisson processes. Oxford Studies in Probability, 3. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1993.

[LLR] M.R. Leadbetter, G. Lindgren, H. Rootzén. Extremes and Related Properties of Random Sequences and Processes, Springer, Berlin-Heidelberg-New York, 1983.

[L] M. Ledoux. On the distribution of overlaps in the Sherrington-Kirkpatrick spin glass model. J. Statist. Phys. 100 (2000), 871-892.

[LT] M. Ledoux, M. Talagrand. Probability in Banach spaces. Springer, Berlin-Heidelberg-New York, 1991.

[MPV] M. Mézard, G. Parisi, M.A. Virasoro. Spin-glass theory and beyond. World Scientific, Singapore (1988).

[NS] C.M. Newman, D.L. Stein. Thermodynamic chaos and the structure of short-range spin glasses. Mathematical aspects of spin glasses and neural networks, 243-287, Progr. Probab., 41, Birkhäuser Boston, Boston, MA, 1998.

[OP] E.Olivieri, P. Picco. On the existence of thermodynamics for the random energy model. Comm. Math. Phys. 96 (1984), 125-144.

[Re] S.I. Resnick, Extreme values, regular variation , and point processes, Springer, New York, 1987.

[Ru] D. Ruelle. A mathematical reformulation of Derrida's REM and GREM. Math. Phys 108 (1987), 225-239 .

[SK] D. Sherrington, S. Kirkpatrick. Solvable model of a spin glass. Phys. Rev. Lett. 35 (1972), 1792-1796.

[T1] M. Talagrand. The Sherrington-Kirkpatrick model: A challenge for mathematicians. Probab. Theory Related Fields 110 (1998), 109-176.

[T2] M. Talagrand. Rigorous results for the Hopfield model with many patterns. Probab. Theory Related Fields 110 (1998), 177-276.

[T3] M. Talagrand. Rigorous low-temperature results for the mean field $p$-spins interaction model. Probab. Theory Related Fields 117 (2000), 303-360.

[T4] M. Talagrand. On the $p$-spin interaction model at low temperature. C. R. Acad. Sci. Paris Sér. I Math. 331 (2000), 939-942.

[T5] M. Talagrand. Mean field models for spin glasses: a first course. Course given in Saint Flour (2000).

[T6] M. Talagrand. Self organization in the low temperature region of a spin glass model. Preprint (2002).