On the Gibbs phase rule in the Pirogov-Sinai regime

A. Bovier, I. Merola, E. Presutti, and M. Zahradník

Abstract. We consider extended Pirogov-Sinai models including lattice and continuum particle systems with Kac potentials. Call λ an intensive variable conjugate to an extensive quantity α appearing in the Hamiltonian via the additive term $-\lambda\alpha$. We suppose that a Pirogov-Sinai phase transition with order parameter α occurs at $\lambda = 0$, and that there are two distinct classes of DLR measures, the plus and the minus DLR measures, with the expectation of α respectively positive and negative in the two classes. We then prove that $\lambda = 0$ is the only point in an interval I of values of λ centered at 0 where this occurs, namely the expected value of α is positive, respectively negative, in all translational invariant DLR measures at $\{\lambda > 0\} \sqcap I$ and $\{\lambda < 0\} \sqcap I$.

1. Introduction

"In the abstract space of all potentials, phase transitions are an exception". This statement by Ruelle in his classical textbook, [**10**], suggests the validity of the Gibbs phase rule, but the notion must be accepted only very cautiously, as remarked by van Enter, [**5**], [**14**] and then by Sokal and Israel, [**13**], [**7**]; anyway a complete proof of the Gibbs rule would require to show that in the space of the few

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thermodynamically relevant parameters, phase transitions occur on regular manifolds of positive co-dimension. But, as stated again by Ruelle in a recent review on open problems in mathematical physics, [**11**], the proof of such a statement must be regarded as one of the main challenges in statistical mechanics.

In the Pirogov-Sinai regime where configurations can be described by contours which satisfy Peierls conditions, the situation is definitely better, as the theory provides tools for a very detailed knowledge on the structure of Gibbs measures in a region in the relevant parameters space; see for instance Sinai's book on phase transitions, [**12**]. The traditional Pirogov-Sinai theory is a low temperature expansion which enables to control the entropic fluctuations from the ground states, its natural setup being the lattice systems. But the theory is not limited to such cases and it has been applied to a great variety of situations, covering various types of phase transitions. We just mention here the case of Kac potentials, which are seen as a perturbation of mean field, where the small parameter is the inverse interaction range of a Kac potential. According to van der Waals, the theory becomes then well suited for investigating the liquid-vapor branch of the phase diagram and, as shown in [**9**], its applications are not restricted to lattice models, [**4**], [**2**], but continuum particle systems can be treated as well.

All the above cases have a common structure. There is a term in the Hamiltonian of the form $-\lambda\alpha$, where α is an extensive quantity and $\lambda \in \mathbb{R}$ is its conjugate variable: in the case of spins λ is an external magnetic field and α the spin magnetization; for particles, λ is the chemical potential and α the particles number. Our main assumption is that at a value, say $\lambda = 0$, of the intensive parameter there is phase coexistence with α an order parameter, and that defining contours in terms of the variable α, the contours satisfy the Peierls bounds with suitable coefficients. Under this assumption the process described in terms of the variable α has the typical features of a low temperature Ising model. We will thus have a class of "plus" measures where α is typically positive (as well as its expectation) and a class of "minus" measures with α typically negative. We are talking of classes of plus and minus measures and not just of plus and minus measures, because we are not ruling out the possibility of other phase transitions, described by other order parameters. As a trivial example, just take the direct product with an Ising model which has a phase transition, thus doubling the extremal DLR measures. In conclusion our assumption implies that at $\lambda = 0$ there are two distinct classes of DLR measures, the plus and minus ones, for which the expected value of α is positive, respectively negative, and which are obtained by thermodynamic limits with plus, respectively minus, boundary conditions. Under this assumption (plus some technical conditions of super-stability type if the variables are unbounded) we prove that there is a finite interval I of values of λ, centered at $\lambda = 0$, where coexistence occurs only at $\lambda = 0$. More precisely, if $\lambda > 0$ (or $\lambda < 0$) and in I, then any translational invariant DLR measure has positive (negative) expectation, and both plus and minus boundary conditions produce in the thermodynamic limit the same class of states.

The proof does not need to specify the precise structure of α, nor the precise definition of contours and it covers in a unified way both perturbations of ground states and Kac potentials. The proof is truly simple, at least we hope, and it avoids the traditional requirements about the structure of the restricted ensembles free energies, which in the applications are not always easy to verify. A more detailed (and maybe more model dependent) analysis is needed if we want to investigate finer questions as differentiability or analytic continuation. As shown recently by Friedli and Pfister[**6**], also for such questions the analysis extends from the traditional setup of the Pirogov-Sinai regime to models with Kac potentials.

In Section 2 we specify the setup and state the main results. In Section 3 we recall the Peierls argument to prove the occurrence of a phase transition at $\lambda = 0$. Section 4 is the most original part of the paper, we prove there that if λ is positive or negative, then, correspondingly, the plus or the minus phases are stable, namely Peierls bounds hold true for the corresponding contours. In Section 5, we complete the analysis by showing uniqueness. In Section 6 we prove that the assumptions in Section 2 are verified in the LMP model for phase transitions in the continuum.

2. Model and main results

State space

After partitioning the physical space ($\mathbb{Z}^d$ or $\mathbb{R}^d$, $d \geq 2$) into cells made of cubes of side ℓ (ℓ a free parameter whose choice will be discussed later), the system becomes a lattice model with a general single site state space Q. In this way we will treat in a uniform way models on a lattice and in the continuum. The main examples we have in mind are finite spins and point particles:

$$Q := \begin{cases} \{-k, ..., k\}^{\ell^d} & \text{with } k \text{ and } \ell \text{ positive integers} \\ \\ \otimes_{n=0}^{\infty} \left(\ (-\ell/2, \ell/2]^d \ \right)^n & n \in \mathbb{N}, \ell \in \mathbb{R}_+ \end{cases}$$

A configuration is denoted by $q := \{q_x\}_{x \in \mathbb{Z}^d}$, which is an element of the phase space $Q^{\mathbb{Z}^d}$, q_x being the configuration at $x \in \mathbb{Z}^d$. In the case of particles, $q_x := (n; r_1, \ldots, r_n)$ is the collection of positions of the particles which are in the cell C_x. We equip the space $Q^{\mathbb{Z}^d}$ with the product topology corresponding to the metric topology on Q, and we suppose that if Q is not compact there is a non negative function $|q|$ on Q such that for any $t \geq 0$, $\{|q| \leq t\}$ is a compact subset of Q. In the first case, we denote by $|q|$ the ℓ_1 -norm on $\mathbb{N}^{\ell^d}$ and in the second case, we define $|q| = |(n, r_i, \ldots, r_n)| \equiv n$ to be the number of particles in the configuration Q.

The space Q is equipped with an a-priori measure ν, which in the first case is just the uniform measure on $\{-k, ..., k\}^{\ell^d}$, an in the case of particles is $\frac{1}{n!}$ times the Lebesgue measure on $(-\ell/2, \ell/2]^{dn}$. We denote by $\nu_\Lambda(dq_\Lambda)$ the product measures $\prod_{x \in \Lambda} \nu(dq_x)$.

In the sequel we will call two sites, x and y, connected if $|x - y| \leq 1$, thus in this paper connected sets are what is often called $*$ connected sets.

Free measure

We consider a fixed probability measure on Q (the counting measure or the law of a Poisson process, in above examples). The free measure is then the product measure ν on $Q^{\mathbb{Z}^d}$. $\nu_\Lambda(\cdot)$ denotes the marginal of $\nu(dq)$ on Q^Λ for $\Lambda \subset \mathbb{Z}^d$.

The models we consider are defined by Hamiltonians obtained in the usual way from interaction potentials. We will always assume translational invariance, finite range and continuity of the potentials, which are then bounded when Q is compact. In the non-compact case, we allow for unbounded potentials, supposing they give rise to Gibbs measures μ with the property that the interactions remain almost surely bounded.

Unperturbed Hamiltonian and contours

We fix a reference Hamiltonian H_0 which incorporates the inverse temperature β as a factor, so that H_0 is dimensionless. We define H_0 by giving the family of its potentials $\{U_\Delta\}$, recalling that the energy of a configuration q_Λ in the finite region Λ in interaction with the configuration p in the complement of Λ is given by the formula

$$H_0(q_\Lambda|p_{\Lambda^c}) \equiv \sum_{\Delta\sqcap\Lambda\neq\emptyset} U_\Delta(q_\Lambda, p_{\Lambda^c})$$

where $(q_\Lambda, p_{\Lambda^c})$ is the configuration which agrees with q_Λ and p_{Λ^c} respectively on Λ and Λ^c. We choose the cell length ℓ in such a way that $U_\Delta \equiv 0$ unless Δ is a translate of the cube $|x| \leq 1$. Abusing notation we then write U_x for the potential relative to the cube Δ with center at x. We then have

$$H_0(q_\Lambda|p_{\Lambda^c}) = \sum_{x:\text{dist}(x,\Lambda)\leq 1} U_x\big((q_\Lambda, p_{\Lambda^c})\big) \tag{2.1}$$

We suppose that $U_x(q)$ is a translation invariant function, i.e.

$$U_x(q) = U_0(\tau_x q), \qquad (\tau_x q)_y = q_{x+y}$$

with $U_0(q)$ depending only on $\{q_y, |y| \leq 1\}$. The other assumptions on H_0 will be stated later.

We are going to describe the configurations in terms of contours, and, as we will see later, the validity of a Peierls condition for H_0 will constitute the main assumption on the Hamiltonian. By adjusting the choice of ℓ, contours are simply described in the following way. There is a single spin function $\theta(q_x)$ with values in $\{0, \pm 1\}$, so that the phase variable

$$\Theta(q;x) = \prod_{y:|y-x|\leq 1} \mathbf{1}_{\theta(q_y)=1} - \prod_{y:|y-x|\leq 1} \mathbf{1}_{\theta(q_y)=-1} \tag{2.2}$$

indicates the spatial distribution of phases: namely the regions $\{\Theta(q;x) = \pm 1\}$ are respectively called the $\pm$ equilibrium regions, while the maximal connected components of $\{\Theta(q;x) = 0\}$, denoted by $\mathrm{sp}(\Gamma_i)$, are the spatial supports of the contours. A contour Γ is then the pair $\Gamma = (\ \mathrm{sp}(\Gamma), \theta_\Gamma)$, where θ_Γ is the restriction of θ to $\mathrm{sp}(\Gamma)$.

In the sequel all configurations q that appear as boundary conditions will be assumed to be such that the set of points $x \in \mathbb{Z}^d$ where $\theta(q_x) \neq 0$ is compact.

The full Hamiltonian

The full Hamiltonian that we consider has the expression:

$$H_\lambda(q) = H_0(q) + \lambda H_I(q) \tag{2.3}$$

where λH_I is the perturbation and λ the coupling strength. The main examples we have in mind are when λ is a magnetic field and $-H_I(q)$ the magnetization density, and when λ is the chemical potential while $-H_I(q)$ is the particles density. More generally and analogously to (2.1) we set

$$H_I(q_\Lambda | p_{\Lambda^c}) = -(\beta\ell^d) \sum_{x:\mathrm{dist}(x,\Lambda)\le 1} \alpha_x((q_\Lambda, p_{\Lambda^c})) \tag{2.4}$$

with $\alpha_x(q) = \alpha_0(\tau_x q)$ dependent only on q_y, $|y - x| \le 1$ (after a proper choice of the cell length ℓ). Thus, in the above examples, $\alpha_x(q)$ is respectively the magnetization and the particles densities in the cell indexed by x.

The coefficient $\beta\ell^d$ could be incorporated in λ, but we prefer to have the two quantities separated, to keep a connection with the original model, before its representation in cells. We will suppose that $\alpha_x(q)$ is bounded on the sets $\{|q_y| \le t,\ |y - x| \le 1\}$ for any $t \ge 0$. In particular $\alpha_x(q)$ is bounded if Q is compact.

We will suppose that α is an order parameter for H_0, namely that there are $\alpha_+ > \alpha_-$ so that

$$\Theta(q;x) = \pm 1 \implies \big|\alpha_x(q) - \alpha_\pm\big| \le \zeta, \quad \zeta = \frac{\alpha_+ - \alpha_-}{10} \tag{2.5}$$

(any other choice with $\zeta < (\alpha_+ - \alpha_-)$ would work as well).

Gibbs and DLR measures

The Gibbs measure $\mu^{(p)}_{\lambda,\Lambda}(dq)$ on the finite region $\Lambda \subset \mathbb{Z}^d$ with b.c. p is given by

$$\mu^{(p)}_{\lambda,\Lambda}(dq) := \frac{e^{-H_\lambda(q_\Lambda|p_{\Lambda^c})}}{Z^{(p)}_{\lambda,\Lambda}} \nu_\Lambda(dq_\Lambda)\delta_{p_{\Lambda^c}}(dq_{\Lambda^c}) \tag{2.6}$$

The denominator $Z^{(p)}_{\lambda,\Lambda}$ is the partition function.

$$Z^{(p)}_{\lambda,\Lambda} := \int \nu_\Lambda(dq_\Lambda)\delta_{p_{\Lambda^c}}(dq_{\Lambda^c})e^{-H_\lambda(q_\Lambda|p_{\Lambda^c})} \tag{2.7}$$

and

$$H_\lambda(q_\Lambda|p_{\Lambda^c}) = \sum_{x:\mathrm{dist}(x,\Lambda)\le 1} \Big(U_x\big((q_\Lambda,p_{\Lambda^c})\big) - \lambda(\beta\ell^d)\alpha_x\big((q_\Lambda,p_{\Lambda^c})\big)\Big)$$

A DLR measure $\mu(dq)$ is then defined by the local specifications $\mu^{(p)}_{\lambda,\Lambda}(dq)$, by requiring that the conditional probability of $\mu(dq)$ on the σ-algebra $\mathcal{F}_{\Lambda^c}$ generated by $\{q_x, x \in \Lambda^c\}$, is μ a.s. equal to $\mu^{(p)}_{\lambda,\Lambda}(dq)$.

The $\pm$ Gibbs measures.

The configuration p is a $\pm$ boundary condition w.r.t. the finite region $\Lambda \sqsubset \mathbb{Z}^d$, if $\theta(p_x) \equiv \pm 1$ for all $x \in \Lambda^c$ such that $\mathrm{dist}(x,\Lambda) \in \{1,2\}$. Then the $\pm$ finite volume, Gibbs measures $\mu^{(\pm,p)}_{\lambda,\Lambda}$ are

$$\mu^{(\pm,p)}_{\lambda,\Lambda}(dq) := \frac{e^{-H_\lambda(q_\Lambda|p_{\Lambda^c})}}{Z^{(\pm,p)}_{\lambda,\Lambda}} \mathbf{1}_{\{\Theta(q,x)=\pm 1,\ \mathrm{dist}\ (x,\Lambda)=1\}}\ \nu_\Lambda(dq_\Lambda)\delta_{p_{\Lambda^c}}(dq_{\Lambda^c}) \tag{2.8}$$

where the partition function $Z^{(\pm,p)}_{\lambda,\Lambda}$ is defined as the normalization factor for $\mu^{(\pm,p)}_{\lambda,\Lambda}(dq)$ to be a probability.

The Peierls condition

We will say that the Peierls condition holds with constant $C_P > 0$, P standing for Peierls, if for any finite region Λ, any $\pm$ boundary condition p and any sequence $\Gamma_1,\dots,\Gamma_m$ of contours with $\mathrm{sp}(\Gamma_i) \sqsubset$

Λ,

$$\mu^{(\pm,p)}_{0,\Lambda}\Big(q : \Gamma_1,\dots,\Gamma_m \text{ are contours of } q\Big) \le e^{-C_P \sum_i |\Gamma_i|} \tag{2.9}$$

where we have denoted by $|\Gamma| := |\mathrm{sp}(\Gamma)|$.

Notice that the condition refers only to $\lambda = 0$. It will be used under the assumption that C_P is large: that is, C_P larger than some universal, dimension dependent constant. Under such an assumption, (2.9) implies the occurrence of a phase transition, as stated in Theorem 2.1.

The condition C_P large is in general model dependent, and our point here is that once and for whatever reason, the condition is checked to hold, then all the properties we will state below are valid and can be applied.

In traditional Pirogov-Sinai models the condition that C_P is large is verified when β is large, while in the case of Kac potentials it follows from assuming the scaling parameter γ small. In both cases $(\beta\ell^d)$ large is the condition responsible for the largeness of C_P. While C_P diverges with $(\beta\ell^d)$, in general the ratio $C_P/(\beta\ell^d)$ is infinitesimal and the range of values of the coupling constant λ will be determined by the ratio:

$$\delta := \frac{C_P}{\beta\ell^d} \tag{2.10}$$

Temperedness conditions

Let $\{\epsilon_i\}$ parameters with values in ±1, we denote by $\mu^{(\{\epsilon_i\},p)}_{\lambda,\Lambda}$ the measure $\mu^p_{\lambda,\Lambda}$ conditioned on $\{\Theta(\cdot;x) = \epsilon_i \text{ on } T_i \sqcap \mathrm{dist}(x,\Lambda) = 1\}$ and suppose that there are constants c_n, c'_n, $n \in \mathbb{N}$, so that

$$\mu^{(\{\epsilon_i\},p)}_{\lambda,\Lambda}\big(|\alpha_x(q)|^n\big) \le c_n, \qquad \text{for any } x \in \Lambda \tag{2.11}$$

$$\mu^{(\{\epsilon_i\},p)}_{\lambda,\Lambda}\Big(q : |q_x| \ge t\Big) \le c'_n t^{-n}, \quad \text{for any } x \in \Lambda \text{ and } t > 1 \tag{2.12}$$

and call "tempered" a probability on $Q^{\mathbb{Z}^d}$ such that, for some coefficient c'_n,

$$\mu\Big(q : |q_x| \ge t\Big) \le c'_n t^{-n}, \quad \text{for any } x \in \mathbb{Z}^d \text{ and } t > 1 \tag{2.13}$$

Large deviations, Cut and paste bounds

There is a positive function $C^*(a)$, $a > 0$, divergent as $a \to \infty$, with the following property. Let Λ be any finite region, $\{T_i\}$ the maximal, connected components of the complement of Λ, Δ any subset of Λ, then:

$$\mu_{\lambda,\Lambda}^{(\{\epsilon_i\},p)}\big(|\alpha_\Delta(q)| \geq a|\Delta|\big) \leq e^{-(\beta\ell^d)C^*(a)|\Delta|}, \qquad \alpha_\Delta = \sum_{x\in\Delta} \alpha_x \quad (2.14)$$

Notice that the above condition is automatically satisfied when Q is compact and α_x is bounded.

In Section 5 (when proving that plus and minus Gibbs states have same thermodynamic limit), we will need the following cut and paste bound, namely that there are a constant $c > 0$ and $n > 0$ so that the following holds. Let Δ be a bounded region, $t > 1$, $\bar{q}$ and p two configurations such that, for any x, $|\bar{q}_x| \leq t$ and $|p_x| \leq t$. Then, for $\epsilon_0 = \pm 1$,

$$\frac{Z_{\lambda,\Delta}^{(\bar{q})}\big(|q_x| \leq t, x \in \Delta\big)}{Z_{\lambda,\Delta}^{(p)}\big(\theta(q_x) = \epsilon_0, x \in \Delta\big)} \;\leq\; e^{c(\beta\ell^d)t^n|\Delta|} \qquad (2.15)$$

where the argument of the partition function means that the partition function is defined with the constraint indicated by the argument.

The bound (2.15) could be proved as a consequence of assumptions on the interaction like

$$\big|H_{\lambda,\Delta}(q_\Delta|\bar{q}_{\Delta^c})\big|,\; \big|H_{\lambda,\Delta}(q_\Delta|p_{\Delta^c})\big| \leq C_I t^n|\Delta| \qquad (2.16)$$

(C_I a suitable constant, proportional to β, while n is determined by the form of the interaction: e.g. $n = 2$ for pair interactions, $n = 4$ in the LMP model) and on assumptions on the free measure of the sets $\theta = \pm 1$:

$$\int_{\theta(q_x)=\epsilon_0} \nu_x(dq_x) \geq e^{-c\ell^d} \qquad (2.17)$$

The sets $\mathcal{G}_\lambda^\pm$

For any bounded set Λ in $\mathbb{Z}^d$, we denote by $\mathcal{G}_{\lambda,\Lambda}^+$ the weak closure of the family of measures which are convex combinations of the family $\{\mu_{\lambda,\Lambda_i}^{(+,p_i)}, \Lambda_i \sqsupseteq \Lambda\}$. $\mathcal{G}_{\lambda,\Lambda}^-$ is defined analogously. Then, for any

$\mu \in \mathcal{G}^{\pm}_{\lambda,\Lambda}$,

$$\mu\Big(q : |q_x| \ge t\Big) \le c'_n t^{-n}, \quad \text{for any } x \in \Lambda \text{ and } t > 1$$

(c'_n as in (2.12)). As a consequence, $\mathcal{G}^{\pm}_{\lambda}$ is weakly compact and since

$$\mathcal{G}^{\pm}_{\lambda,\Lambda} \sqsupseteq \mathcal{G}^{\pm}_{\lambda,\Delta}, \quad \text{when } \Delta \sqsupseteq \Lambda$$

we conclude that

$$\mathcal{G}^{+}_{\lambda} = \sqcap\, \mathcal{G}^{+}_{\lambda,\Lambda}, \qquad \mathcal{G}^{-}_{\lambda} = \sqcap\, \mathcal{G}^{-}_{\lambda,\Lambda}$$

are both non empty, weakly compact, convex sets of tempered DLR measures. Notice that $\mathcal{G}^{\pm}_{\lambda}$, which are both non empty, are however not necessarily distinct. If this is the case, then indeed there is a phase transition, otherwise $\mathcal{G}^{+}_{\lambda} = \mathcal{G}^{-}_{\lambda}$. Take notice that this does not mean uniqueness of DLR measures, as there could be several DLR measures in the plus and/or minus phases.

We can now state our main theorem.

Theorem 2.1. *There are dimensions dependent constants $\hat{\lambda}$, $C_P(d)$, $d \ge 2$, so that if $C_P \ge C_P(d)$, then:*

- *At $\lambda = 0$, $\mathcal{G}^{+}_{0} \sqcap \mathcal{G}^{-}_{0} = \emptyset$ and if $\mu \in \mathcal{G}^{\pm}_{0}$ then, for all x, $\mu(\Theta(\cdot;x) = \pm 1) > 1/2$.*

If, moreover, there is $a^* > 0$ so that $(\beta\ell^d)C^*(a^*) > C_P$ (in the sequel we shorthand $C^* = C^*(a^*)$, the latter defined in (2.14)), then for suitable constants $C(d) > 0$:

- *If $0 < \lambda \le \lambda^* = \min\{C(d)\delta^2, \hat{\lambda}\}$, (resp. $0 > \lambda \ge -\lambda^*$), any tempered DLR measure μ (see (2.13)) is in $\mathcal{G}^{+}_{\lambda}$ (resp. $\mathcal{G}^{-}_{\lambda}$) and, for all x, $\mu(\Theta(\cdot;x) = 1) > 1/2$ (resp. $\mu(\Theta(\cdot;x) = -1) > 1/2$).*

Theorem 2.1 will be proved in the remaining sections, in particular the statement at $\lambda = 0$ is proved in Section 3, while the statements at $\lambda \ne 0$ are proved in Sections 4 and 5.

Theorem 2.1 proves uniqueness in a "small" interval of values of λ, in particular in the case of Kac potentials the interval becomes infinitesimal as $\gamma \to 0$, γ the Kac scaling parameter, see Section 6 for the LMP model. A different argument must be used for the larger λ, which in many applications (including Kac potentials and traditional Pirogov-Sinai models) is based on the fact that the double well structure (of the energy or of the mean field free energy, for Kac potentials) at $\lambda = 0$ is unbalanced when $\lambda \neq 0$. The one well case is in general "easy to treat" at least if the unbalance between wells is not too small, i.e. if λ is large enough. Thus our theorem covers the dangerous case of the small perturbations. The extension to larger λ involves new assumptions on the structure of the Hamiltonian and seems more model dependent. In separate papers some specific cases will be treated.

3. Phase transitions and the Peierls argument

In this section we will prove the statement of Theorem 2.1 about the occurrence of a phase transition at $\lambda = 0$. The proof is nothing but the classical Peierls argument and we report it mainly because it introduces notions that will be used in the sequel.

Geometry of contours.

Given a contour Γ, we denote by $\text{ext}(\Gamma)$ the unbounded maximal connected component of $\text{sp}(\Gamma)^c$. The contour is called a $+$ $[-]$ contour if $\theta_\Gamma(x) = 1$ [resp. $= -1$] for all x such that $\text{dist}(x,\text{ext}(\Gamma)) = 1$. By definition of contours θ_Γ is constant on $\{x : \text{dist}(x,\text{ext}(\Gamma)) = 1\}$ and different from 0. We call $\text{int}(\Gamma) = \text{sp}(\Gamma)^c \setminus \text{ext}(\Gamma)$, and distinguish $\text{int}^\pm(\Gamma)$ as the union of all the maximal connected components of $\text{int}(\Gamma)$ which are connected to regions where θ_Γ is respectively ± 1. Finally we call

$$V(\Gamma) = \text{sp}(\Gamma) \sqcup \text{int}(\Gamma) \tag{3.1}$$

Mass of contours.

The mass of a $\pm$ contour Γ relative to a probability μ on $Q^{\mathbb{Z}^d}$, is defined for all configurations p which are respectively $\pm$ b.c. relative to the region $V(\Gamma)$, as

$$\mu(\Gamma;p) := \mu\Big({\scriptstyle q:\Gamma \text{ is a } \pm \text{ contour for } q} \;\Big|\; {\scriptstyle \mathcal{F}_{V(\Gamma)^c};\{\Theta(q;x)=\pm 1, \mathrm{dist}(x,V(\Gamma))=1\}} \Big)(p) \tag{3.2}$$

namely $\mu(\Gamma;p)$ is the conditional probability computed at p of having the contour Γ, given the σ-algebra generated by the variables q_x, $x \in V(\Gamma)^c$ and given that $\{\Theta(q;x) = \pm 1, \mathrm{dist}(x, V(\Gamma)) = 1\}$.

If Γ is a $\pm$ contour and $\mu = \mu^{(\pm,p)}_{\lambda,V(\Gamma)}$, we write its mass as $m_\lambda(\Gamma;p)$, namely

$$m_\lambda(\Gamma;p) := \mu^{(\pm,p)}_{\lambda,V(\Gamma)}\Big(q : \Gamma \text{ is a contour for } q\Big) \tag{3.3}$$

which, sometimes, will be simply called "the mass of the contour Γ".

Notice that a different notion is often used in the literature, namely that of "the weight of a contour", which is defined in terms of a ratio of two partition functions, one with the constraint of having a contour Γ and the other one with the constraint that the contour Γ is absent.

Finally, μ satisfies the Peierls condition relative to a class $\{\Gamma\}$ of contours and with constant C, if

$$\mu(\Gamma;p) \le e^{-C|\Gamma|} \text{ for all contours } \Gamma \in \{\Gamma\} \text{ and all } p \tag{3.4}$$

Lemma 3.1. *If C is large enough, then*

$$\sum_{\Gamma:0\in V(\Gamma)} e^{-C|\Gamma|} \;\le\; e^{-C} \tag{3.5}$$

The proof starts from the observation that $|\Gamma| \ge 9$ (in $d = 2$ the minimal contour consists of a site and those connected to it). Thus the sum in (3.5) starts from sets with at least 9 sites. Since the number of contours Γ whose support contains the origin with $|\Gamma| = M$ is bounded by K^M for some finite constant K, we see that

$$\sum_{\Gamma:0\in V(\Gamma)} e^{-C|\Gamma|} \le \sum_{M=9}^{\infty} K^M e^{-CM} \le \frac{(Ke^{-C})^9}{1-Ke^{-C}}$$

which implies 3.5 if C is sufficiently large.

Proof of the statement in Theorem 2.1 at $\lambda = 0$.

Suppose the measures $\mu_{0,\Lambda}^{(\pm,p)}$ satisfy (3.4) with $C > 1$ and as large as required by Lemma 3.1. Then, for any $x \in \Lambda$:

$$\mu_{0,\Lambda}^{(\pm,p)}\big(|\alpha_x - \alpha_\pm| > \zeta\big) = \mu_{0,\Lambda}^{(\pm,p)}\big(x \in V(\Gamma), \text{for some } \Gamma\big) \leq 1/e \quad (3.6)$$

The inequality remains valid after convex combinations and weak limits, proving that $\mathcal{G}_0^{\pm}$ are distinct.

Some generalizations.

In the next section we will use variants of the above argument which are stated below. We will denote by $\mu(f)$ the expectation of a function f, relative to a measure μ.

Lemma 3.2. *Let μ be supported by configurations q having $\Theta(q;x) = \epsilon_0$, $\epsilon_0 \in \{\pm 1\}$, for all $x \in \Lambda^c$, Λ a bounded region. Then, if μ satisfies the Peierls bounds for all ϵ_0-contours with a constant C as large as needed in Lemma 3.1, for any $x \in \Lambda$*

$$\mu\big(|\alpha_x - \alpha_{\epsilon_0}|\big) \leq \zeta + e^{-C/2}\, \mu\Big(|\alpha_x - \alpha_{\epsilon_0}|^2\Big)^{1/2} \qquad (3.7)$$

$$\mu\big(|\alpha_x - \alpha_{\epsilon_0}| \mathbf{1}_{|\alpha_x - \alpha_{\epsilon_0}| > \zeta}\big) \leq e^{-C/2}\, \mu\Big(|\alpha_x - \alpha_{\epsilon_0}|^2\Big)^{1/2} \qquad (3.8)$$

Proof. We partition the space into the sets $|\alpha_x - \alpha_{\epsilon_0}| \leq \zeta$ and $|\alpha_x - \alpha_{\epsilon_0}| > \zeta$. The contribution to the expectation (3.7) of the first set is bounded by ζ. All configurations in the support of μ which belong to the second set, have a ϵ_0-contour Γ with $x \in V(\Gamma)$. We then use Cauchy-Schwartz and (3.5) to prove (3.7) and (3.8). $\square$

4. Stability of phases

In this section we will prove that for $0 < \lambda \le \lambda^*$,

$$m_\lambda(\Gamma; q) \le e^{-(C_P/8)|\Gamma|}, \quad \text{for any } + \text{ contour } \Gamma \tag{4.1}$$

and that the analogous property holds for negative λ and $-$ contours.

By standard arguments and using the analysis of Section 3 (details are omitted), it follows from (4.1) that for C_P large enough, if $\lambda > 0$ [$\lambda < 0$], then any measure in $\mathcal{G}^+$ [in $\mathcal{G}^-$] is supported by configurations q such that the set $\{x : \Theta(q, x) = 1\}$ [$\{x : \Theta(q, x) = -1\}$] occupies most of the space. The persistence of this property past $\lambda = 0$ is usually referred to as "a stability of the + (or $-$) phase " under the perturbation of strength λ.

To have lighter notation we restrict in the sequel to positive λ, the proofs for negative λ being completely similar are omitted. (4.1) will follow from showing that the typical configurations of $\mu^{(-,p)}_{\lambda,\Lambda}$ have a layer close to the boundary of Λ, where mostly $\Theta = -1$, but after the layer typically $\Theta = 1$. An improved control on the thickness of the layer and on the absence of finger-like protuberances will prove in Section 5 that any tempered DLR measure is in $\mathcal{G}^+$.

The key point is to show that $\Theta = -1$ has a cost and that it is therefore convenient to have a transition to $\Theta = 1$. The cost comes from the energy $-\lambda\alpha(q)$ which would decrease from erasing the minus layer and thus changing $\Theta = -1$ into $\Theta = 1$. The problem comes from the entropic factor due to the number of ways to realize the minus layer attached to the boundary of Λ. If λ is very small, such an entropic factor becomes too large and the layer impossible to erase. We will fix the problem by not regarding as being in the plus phase any region with $\Theta = 1$ whose volume is "too small": the entropy is then drastically reduced and under control. With this in mind we introduce the notion of

"Slim" and "Fat" contours

Slim and fat contours are distinguished using the length scale

$$\xi_-(\lambda) := \lambda^{-1/2} \tag{4.2}$$

recall that we are supposing $\lambda > 0$ small and therefore $\xi_-(\lambda)$ is a large quantity. The choice of the power $1/2$ in the above definition is to a large extent arbitrary.

Definition 4.1. *A contour Γ is called "λ-slim" if*

$$|V(\Gamma)| \leq \xi_-(\lambda)^d \tag{4.3}$$

where $V(\Gamma)$ is defined in (3.1)*. A contour that is not λ-slim is called "λ-fat". Similarly, a region Λ is called "λ-fat" if it is a bounded region, such that all the maximal connected components of Λ^c have volume larger than $\xi_-(\lambda)^d$.*

Remark: When there is no risk of confusion, we will simply say "slim" and "fat" contours.

Proposition 4.2. *For $0 < \lambda \leq \lambda^*$ the mass of any slim contour Γ is bounded by*

$$m_\lambda(\Gamma; q) \leq e^{-(C_P/2)|\Gamma|} \leq e^{-(C_P/4)|\Gamma| - \lambda(\beta\ell^d)(\alpha_+ - \alpha_-)|V(\Gamma)|} \tag{4.4}$$

(the latter inequality will be used at the end of Section 5). The analogous statement holds for $-\lambda^* \leq \lambda < 0$.

Proof. For notational simplicity we will refer to λ positive and $+$ contours. We will denote by $Z(A)$ a partition function with the constraint that the configurations should belong to the set A. We then have for a $+$ contour Γ,

$$m_\lambda(\Gamma; p) = \frac{Z^{(+,p)}_{\lambda,V(\Gamma)}\Big(\Gamma \text{ is a contour}\Big)}{Z^{(+,p)}_{\lambda,V(\Gamma)}} \tag{4.5}$$

Then, by (2.14) and recalling the definition of a^* in Theorem 2.1,

$$Z^{(+,p)}_{\lambda,V(\Gamma)}\Big(\Gamma \text{ is a contour}\Big) \leq$$
$$Z^{(+,p)}_{\lambda,V(\Gamma)}\Big(\Gamma \text{ is a contour}; \, |\alpha_{V(\Gamma)}| \leq a^*|V(\Gamma)|\Big) + e^{-C_P|\Gamma|} Z^{(+,p)}_{\lambda,V(\Gamma)}$$

In the first partition function on the r.h.s. we want to replace λ by 0, in order to use the Peierls bounds valid by assumption at $\lambda = 0$. We have

$$H_\lambda(q_{V(\Gamma)}|q_{V(\Gamma)^c}) - H_0(q_{V(\Gamma)}|q_{V(\Gamma)^c}) = -\lambda(\beta\ell^d) \sum_{\mathrm{dist}(x,V(\Gamma))\leq 1} \alpha_x(q)$$

On the other hand, due to the constraints in the partition function, we can restrict to configurations q which agree with p on Λ^c, and such that both $|\alpha_{V(\Gamma)}| \le a^*|V(\Gamma)|$ and $\Theta(q;x) = 1$ when $\mathrm{dist}(x, V(\Gamma)) = 1$. Calling $|\alpha_\pm| = \max\{|\alpha_+|, |\alpha_-|\}$, we have

$$\begin{aligned} &|H_\lambda(q_{V(\Gamma)}|q_{V(\Gamma)^c}) - H_0(q_{V(\Gamma)}|q_{V(\Gamma)^c})| \le X \\ &X := |\lambda|\{(\beta\ell^d)a^*|V(\Gamma)| + (2|\alpha_\pm|)(2d)|\Gamma|\} \end{aligned} \tag{4.6}$$

because the cardinality of $\{x : \mathrm{dist}(x, V(\Gamma)) = 1\}$ is bounded by $2d|\Gamma|$ and $|\alpha_+ \pm \zeta| \le 2|\alpha_\pm|$. Then

$$\begin{aligned} Z^{(+,p)}_{\lambda,V(\Gamma)}\Big(\Gamma \text{ is a contour}\Big) &\le e^X Z^{(+,p)}_{0,V(\Gamma)}\Big(\Gamma \text{ is a contour}\Big) + e^{-C_P|\Gamma|} Z^{(+,p)}_{\lambda,V(\Gamma)} \\ &\le e^{X - C_P|\Gamma|} Z^{(+,p)}_{0,V(\Gamma)} + e^{-C_P|\Gamma|} Z^{(+,p)}_{\lambda,V(\Gamma)} \end{aligned}$$

and, writing $Z^{(+,p)}_{0,V(\Gamma)} \le Z^{(+,p)}_{0,V(\Gamma)}\Big(|\alpha_{V(\Gamma)}| \le a^*|V(\Gamma)|\Big) + e^{-C_P|\Gamma|} Z^{(+,p)}_{0,V(\Gamma)}$, we get

$$\begin{aligned} &\le \frac{e^{X - C_P|\Gamma|}}{1 - e^{-C_P|\Gamma|}} Z^{(+,p)}_{0,V(\Gamma)}\Big(|\alpha_{V(\Gamma)}| \le a^*|V(\Gamma)|\Big) + e^{-C_P|\Gamma|} Z^{(+,p)}_{\lambda,V(\Gamma)} \\ &\le \Big(\frac{e^{2X - C_P|\Gamma|}}{1 - e^{-C_P|\Gamma|}} + e^{-C_P|\Gamma|}\Big) Z^{(+,p)}_{\lambda,V(\Gamma)} \\ &\le \Big(\frac{e^{2X - C_P|\Gamma|/2}}{1 - e^{-C_P|\Gamma|}} + e^{-C_P|\Gamma|/2}\Big) e^{-C_P|\Gamma|/2} Z^{(+,p)}_{\lambda,V(\Gamma)} \end{aligned}$$

To bound the parameter X introduced in (4.6), we use an isoperimetric inequality: there is $c_d > 0$ such that

$$|V(\Gamma)|^{(d-1)/d} \le c_d|\Gamma|, \qquad |V(\Gamma)| \le c_d|\Gamma|\xi_-(\lambda) \tag{4.7}$$

Then, recalling (2.10),

$$X \le \Big((C_P\delta^{-1}a^*c_d)|\lambda\xi_-(\lambda)| + (2|\alpha_\pm|)(2d)|\lambda|\Big)|\Gamma|$$

By the choice of λ^* in Theorem 2.1, $\lambda\xi_-(\lambda) \le \sqrt{\lambda^*} \le \delta\sqrt{C(d)}$, so that the first inequality in (4.4) follows for $C(d)$ small enough and C_P large enough.

To complete the proof of (4.4) we need to show that

$$\lambda(\beta\ell^d)(\alpha_+ - \alpha_-)|V(\Gamma)| \le (C_P/4)|\Gamma| \tag{4.8}$$

By (4.7),

$$\lambda(\beta\ell^d)(\alpha_+ - \alpha_-)|V(\Gamma)| \le \lambda(\beta\ell^d)(\alpha_+ - \alpha_-)c_d|\Gamma|\xi_-(\lambda)$$

hence (4.8), for $C(d)$ small enough.

□

For later reference we state a property analogous to one established in the proof of Proposition 4.2 and which follows directly from (2.14). Let Γ be a contour and p a configuration which has Γ as a contour. Call $D = \{x : \mathrm{dist}(x, \mathrm{sp}(\Gamma) = 1\}$, then $\theta(p_x)$ is constant on each one of the connected components of D, and we denote by $\epsilon_i = \pm 1$ such values. Then, supposing C_P so large that

$$1 - e^{-C_P} \ge e^{-1}, \qquad 1 + ce^{-C_P/4} \le e \tag{4.9}$$

$$Z_{\lambda,\mathrm{sp}(\Gamma)}^{(\{\epsilon_i\},p)}\Big(|\alpha_{\mathrm{sp}(\Gamma)}| > a^*|\Gamma|\Big) \le e^{-C_P|\Gamma|+1} Z_{\lambda,\mathrm{sp}(\Gamma)}^{(\{\epsilon_i\},p)}\Big(|\alpha_{\mathrm{sp}(\Gamma)}| \le a^*|\Gamma|\Big) \tag{4.10}$$

In fact, by (2.14),

$$Z_{\lambda,\mathrm{sp}(\Gamma)}^{(\{\epsilon_i\},p)}\Big(|\alpha_{\mathrm{sp}(\Gamma)}| > a^*|\Gamma|\Big) \le e^{-C_P|\Gamma|} Z_{\lambda,\mathrm{sp}(\Gamma)}^{(\{\epsilon_i\},p)}$$

We then write

$$Z_{\lambda,\mathrm{sp}(\Gamma)}^{(\{\epsilon_i\},p)} = Z_{\lambda,\mathrm{sp}(\Gamma)}^{(\{\epsilon_i\},p)}\Big(|\alpha_{\mathrm{sp}(\Gamma)}| > a^*|\Gamma|\Big) + Z_{\lambda,\mathrm{sp}(\Gamma)}^{(\{\epsilon_i\},p)}\Big(|\alpha_{\mathrm{sp}(\Gamma)}| \le a^*|\Gamma|\Big)$$

and use (4.9) to get (4.10).

The proof of (4.1) for the fat contours is not as simple as for the slim ones. An important point, see Proposition 4.4 below, is the analysis of the empirical averages of α_x in regions where only slim contours are present and where the boundary conditions are in the minus phase. The setup is described by the measures

$$\mu_{\lambda,\Lambda}^{(-,p,s)}(dq) = \mu_{\lambda,\Lambda}^{(-,p)}\Big(dq \,\Big|\, \{\text{all contours are slim}\}\Big) \tag{4.11}$$

To estimate the contribution of the perturbation λH_I, we will prove that α_Λ is close to $\alpha_-|\Lambda|$, which is not a priori obvious because $\lambda > 0$. After establishing a Peierls bound on contours for the measure $\mu_{\lambda,\Lambda}^{(-,p,s)}$, we will conclude that most of the space is made of sites x with $\Theta(\cdot; x) = -1$ and this will yield the desired result.

Lemma 4.3. *Let Λ be a fat region and $\Delta \sqsubset \Lambda$, $|\Delta| < \xi_-(\lambda)^d$. Then for $\epsilon_0 = \pm 1$,*

$$\mu_{\lambda,\Lambda}^{(-,p,s)}\big(dq|\mathcal{F}_{\Delta^c}, \{\Theta(q;x) = \epsilon_0, dist(x,\Delta) = 1\}\big) = \mu_{\lambda,\Delta}^{(\epsilon_0,p)}(dq) \tag{4.12}$$

As a consequence if Γ *is a slim contour with* $sp(\Gamma) \sqsubset \Lambda$, *then*

$$\mu^{(-,p,s)}_{\lambda,\Lambda}(\Gamma;p) = m_\lambda(\Gamma;p) \tag{4.13}$$

the r.h.s. having been defined in (3.2).

Proof. The measure on the l.h.s. of (4.12) is equal to

$$\mu^{(-,p)}_{\lambda,\Lambda}\Big(dq\big|\mathcal{F}_{\Delta^c},\{\Theta(q;x)=\epsilon_0,\mathrm{dist}(x,\Delta)=1\},\{\text{all contours of } q \text{ are slim}\}\Big)(q')$$

Since $\Theta(q;x) = \epsilon_0$ for all $x : \mathrm{dist}(x,\Delta) = 1$, the contours of q have spatial support either contained entirely in Δ or entirely outside Δ. Since $|\Delta| < \xi_-(\lambda)^d$, we can drop the condition that the contours whose spatial support is in Δ are slim, because it is automatically satisfied, hence (4.12).

If Γ is a slim contour and $\mathrm{sp}(\Gamma) \sqsubset \Lambda$, then $V(\Gamma) \sqsubset \Lambda$ because Λ is fat and (4.13) follows then from (4.12). Lemma 4.3 is proved. □

Proposition 4.4. *There is* $c > 0$ *so that if* $0 < \lambda \le \lambda^*$ *and* Λ *is a fat region, then*

$$\mu^{(-,p,s)}_{\lambda,\Lambda}\Big(q : |\alpha_\Lambda(q) - \alpha_-|\Lambda|| > 2\zeta|\Lambda|\Big) \le ce^{-C_P/4} \tag{4.14}$$

$$\mu^{(+,p)}_{0,\Lambda}\Big(q : |\alpha_\Lambda(q) - \alpha_+|\Lambda|| > 2\zeta|\Lambda|\Big) \le ce^{-C_P/4} \tag{4.15}$$

Proof. We start with (4.15). Since

$$\Big\{|\alpha_\Lambda - \alpha_+|\Lambda|| > 2\zeta|\Lambda|\Big\} \ \sqsubset\ \Big\{\sum_{x\in\Lambda}|\alpha_x - \alpha_+|\mathbf{1}_{|\alpha_x-\alpha_+|>\zeta} > \zeta|\Lambda|\Big\} \tag{4.16}$$

by the Chebishev inequality,

$$\begin{aligned}\mu^{(+,p)}_{0,\Lambda}\Big(q : |\alpha_\Lambda(q) - \alpha_+|\Lambda|| > 2\zeta|\Lambda|\Big) \qquad\qquad (4.17)\\ \le \frac{1}{\zeta}\sup_{x\in\Lambda}\mu^{(+,p)}_{0,\Lambda}\big(|\alpha_x - \alpha_+|\mathbf{1}_{|\alpha_x-\alpha_+|>\zeta}\big)\end{aligned}$$

All q in the support of $\mu^{(+,p)}_{0,\Lambda}$ are such that

$$\text{if } |\alpha_x(q) - \alpha_+| > \zeta \text{ then } q \text{ has a contour } \Gamma \text{ with } x \in V(\Gamma) \tag{4.18}$$

Then, supposing C_P large enough, by Lemma 3.2

$$\mu_{0,\Lambda}^{(+,p)}\Big(q:|\alpha_\Lambda(q)-\alpha_+|\Lambda||>2\zeta|\Lambda|\Big) \le \frac{(e^{-C_P})^{1/2}}{\zeta}\sup_{x\in\Lambda}\mu_{0,\Lambda}^{(+,p)}\big(|\alpha_x-\alpha_+|^2\big)^{1/2} \tag{4.19}$$

(4.15) follows then from (4.19) by using (2.11).

Analogously we have

$$\mu_{\lambda,\Lambda}^{(-,p,s)}\Big(q:|\alpha_\Lambda(q)-\alpha_-|\Lambda||>2\zeta|\Lambda|\Big) \le \frac{1}{\zeta}\sup_{x\in\Lambda}\mu_{\lambda,\Lambda}^{(-,p,s)}\big(|\alpha_x-\alpha_-|\mathbf{1}_{|\alpha_x-\alpha_-|>\zeta}\big) \tag{4.20}$$

For any x on the r.h.s. of (4.20) there is a slim contour Γ such that $V(\Gamma)\ni x$. Then

$$\mu_{\lambda,\Lambda}^{(-,p,s)}\big(|\alpha_x-\alpha_-|\mathbf{1}_{|\alpha_x-\alpha_-|>\zeta}\big) \le \sum_{\substack{\Gamma \text{ is slim}\\ V(\Gamma)\ni x}}\mu_{\lambda,\Lambda}^{(-,p,s)}\Big(|\alpha_x-\alpha_-|\mathbf{1}_{q\to\Gamma}\Big)$$

where $q\to\Gamma$ means that Γ is a contour for q. By (4.12) we get

$$\mu_{\lambda,\Lambda}^{(-,p,s)}\Big(|\alpha_x-\alpha_-|\mathbf{1}_{q\to\Gamma}\Big) = \int \mu_{\lambda,\Lambda}^{(-,p,s)}(dq)\,\mu_{\lambda,V(\Gamma)}^{(\epsilon(\Gamma),q)}\Big(|\alpha_x(q')-\alpha_-|\mathbf{1}_{q'\to\Gamma}\Big)$$

Using (4.4), setting $\epsilon(\Gamma)=\pm1$ if Γ is a $\pm$ contour, we then get

$$\mu_{\lambda,V(\Gamma)}^{(\epsilon(\Gamma),q)}\Big(|\alpha_x(q')-\alpha_-|\mathbf{1}_{q'\to\Gamma}\Big) \le e^{-(C_P/4)|\Gamma|}\mu_{\lambda,V(\Gamma)}^{(\epsilon(\Gamma),q)}\Big(|\alpha_x-\alpha_-|\Big)^{1/2}$$

Using again (2.11), we conclude that if C_P is large enough, then (4.14) holds. Proposition 4.4 is proved. □

The minus layer attached to a plus contour

Resuming the notation (4.5), we fix a fat + contour Γ_0, (Γ_0 because we will use Γ as a current symbol for contours, + because we are proving (4.1) with $\lambda>0$). We also suppose that $\text{int}^-(\Gamma_0)$ is a fat region, in the other case the proof of (4.1) is essentially the same as that of Proposition 4.2 and omitted. Given a configuration q which has Γ_0 as a contour, we select inside $\text{int}^-(\Gamma_0)$ the family $(\Gamma_1,..,\Gamma_n)$ of the fat, minus, "external" contours: these are all the fat, minus contours Γ with $\mathrm{sp}(\Gamma) \sqsubset \mathrm{int}^-(\Gamma_0)$ and which are neither contained in any $V(\Gamma')$, Γ' a plus contour in $\mathrm{int}^-(\Gamma_0)$, nor in the plus interior of any minus contour Γ' in $\mathrm{int}^-(\Gamma_0)$. Thus $(\Gamma_1, .., \Gamma_n)$ are all the fat contours in the subset of $\mathrm{int}^-(\Gamma_0)$ obtained by erasing from the latter all the regions which are surrounded by a circuit with $\Theta = 1$. We will shorthand $\underline{\Gamma} = (\Gamma_0, \Gamma_1, .., \Gamma_n)$, with $(\Gamma_1, .., \Gamma_n)$ as above.

Thus $\underline{\Gamma}$ divides $V(\Gamma_0)$ into three regions:

$$\begin{gathered}
\mathrm{int}^+(\underline{\Gamma}) = \bigsqcup_i \mathrm{int}^+(\Gamma_i), \quad \mathrm{sp}(\underline{\Gamma}) = \bigsqcup_i \mathrm{sp}(\Gamma_i), \\
\Delta^-(\underline{\Gamma}) := V(\Gamma_0) \setminus \Big\{ \mathrm{int}^+(\underline{\Gamma}) \sqcup \mathrm{sp}(\underline{\Gamma}) \Big\}
\end{gathered} \tag{4.21}$$

The minus layer attached to the plus contour Γ_0 is then defined as

$$\Lambda^-(\underline{\Gamma}) = V(\Gamma_0) \setminus \mathrm{int}^+(\underline{\Gamma}) = \mathrm{sp}(\underline{\Gamma}) \sqcup \Delta^-(\underline{\Gamma}) \tag{4.22}$$

Notice that $|\Lambda^-(\underline{\Gamma})| \ge |\Gamma_0|$.

In the next theorem we will prove a bound on the volume of the minus layer $\Lambda^-(\underline{\Gamma})$ which implies (4.1) and which will be crucial in the proof in the next section that all tempered DLR measures are in $\mathcal{G}^+_\lambda$, if $\lambda > 0$ (and in $\mathcal{G}^-_\lambda$, if $\lambda < 0$).

Theorem 4.5. *Let Γ_0 be a fat, plus contour and Λ^- the random set defined in* (4.22)*, then for any $V \ge |\Gamma_0|$,*

$$\begin{gathered}
\mu^{(+,p)}_{\lambda, V(\Gamma_0)} \Big(\exists \underline{\Gamma} : \ |\Lambda^-(\underline{\Gamma})| \ge V \Big) \le e^{-(C_P/8)|\Gamma_0| - \omega V} \\
\omega = \min \{ \frac{C_P}{8} \ ; \ \lambda(\beta \ell^d) \frac{\alpha_+ - \alpha_-}{2} \}
\end{gathered} \tag{4.23}$$

As a consequence

$$m_\lambda(\Gamma_0; p) \le e^{-(C_P/8)|\Gamma_0|} \tag{4.24}$$

Proof. Note first that (4.24) is an obvious consequence of (4.23). Thus we only need to prove the latter inequality.

With the obvious meaning of the symbols, we have

$$\mu^{(+,p)}_{\lambda, V(\Gamma_0)} \Big(|\Lambda^-| \ge V \Big) = \frac{Z^{(+,p)}_{\lambda, V(\Gamma_0)} \Big(|\Lambda^-| \ge V \Big)}{Z^{(+,p)}_{\lambda, V(\Gamma_0)}}$$

To bound the partition function in the numerator we start writing

$$Z^{(+,p)}_{\lambda,V(\Gamma_0)}\Big(|\Lambda^-|\ge V\Big) \le \sum_{\underline{\Gamma}:|\Lambda^-|\ge V} Z^{(+,p)}_{\lambda,V(\Gamma_0)}\Big(q\to\underline{\Gamma}\Big)$$

$$Z^{(+,p)}_{\lambda,V(\Gamma_0)}\Big(q\to\underline{\Gamma}\Big) = \int_{\{p'\to \mathrm{int}^+(\underline{\Gamma})\}} \nu_{\mathrm{int}^+(\underline{\Gamma})}(dp')e^{-H_\lambda(p')}\, Z^{(+,p,p')}_{\lambda,\Lambda^-}\Big(q\to\underline{\Gamma}\Big)$$

where $\{p'\to \mathrm{int}^+(\underline{\Gamma})\}$ is the condition

$$\theta(p'_x)=1 \text{ for all } x\in \mathrm{int}^+(\underline{\Gamma}) : \mathrm{dist}(x,\Lambda^-)\le 2$$

while $\{q\to\underline{\Gamma}\}$ in the last partition function is a shorthand for the condition that the configuration equal to q in Λ^-, to p' in $\mathrm{int}^+(\underline{\Gamma})$ and to p outside $V(\Gamma_0)$ gives rise to $\underline{\Gamma}$.

The rest of the proof will only involve manipulations on $Z^{(+,p,p')}_{\lambda,\Lambda^-}\big(q\to\underline{\Gamma}\big)$. Calling

$$\Delta^-_0=\big\{x\in\Delta^- :\ \mathrm{dist}(x,(\Delta^-)^c)>1\big\},\quad \mathrm{sp}(\underline{\Gamma})^+=\Lambda^-\setminus\Delta^-_0$$

(recall that $\mathrm{sp}(\underline{\Gamma})=\Lambda^-\setminus\Delta^-$), we then have

$$Z^{(+,p,p')}_{\lambda,\Lambda^-}\Big(q\to\underline{\Gamma}\Big) = \int_{\{\theta(q'_x)=\theta_{\underline{\Gamma}}\}} \nu_{\mathrm{sp}(\underline{\Gamma})^+}(dq')e^{-H_\lambda(q'|p,p')}Z^{(-,q',s)}_{\lambda,\Delta^-_0} \tag{4.25}$$

where $\{\theta(q'_x)=\theta_{\underline{\Gamma}}\}$ is a shorthand for the condition that $\theta(q'_x)=\theta_{\Gamma_i}(x)$ when $x\in\mathrm{sp}(\Gamma_i)$, $i=0,..,n$, and moreover that $\theta(q'_x)=-1$ when $x\in\mathrm{sp}(\underline{\Gamma})^+\setminus\,\mathrm{sp}(\underline{\Gamma})$ (which is the same as $\Delta^-\setminus\Delta^-_0$).

By (4.14)

$$Z^{(-,q',s)}_{\lambda,\Delta^-_0}\le\big(1-ce^{-C_P/4}\big)^{-1}Z^{(-,q',s)}_{\lambda,\Delta^-_0}\Big(|\alpha_{\Delta^-_0}-\alpha_-|\Delta^-_0||\le 2\zeta|\Delta^-_0|\Big)$$

which, reinserted into (4.25), supposing C_P so large that $(1-ce^{-C_P/4})^{-1}\le e$, yields:

$$\begin{aligned} Z^{(+,p,p')}_{\lambda,\Lambda^-}\Big(q\to\underline{\Gamma}\Big)\le\ & e\int_{\left\{\substack{|\alpha_{\Delta^-_0}-\alpha_-|\Delta^-_0||\le 2\zeta|\Delta^-_0|,\\ \theta(q_x)=-1,\ x\in\Delta^-\setminus\Delta^-_0}\right\}} \nu_{\Delta^-}(dq')\ e^{-H_\lambda(q'|p,p')} \\ & \times\prod_{i=0}^n\int_{\left\{\substack{\theta(q_x)=\theta_{\Gamma_i}(x),\\ x\in\mathrm{sp}(\Gamma_i)}\right\}} \nu_{\mathrm{sp}(\underline{\Gamma}_i)}(dq)\ e^{-H_\lambda(q|q',p.p')} \end{aligned} \tag{4.26}$$

In each one of the integrals in the above product we distinguish whether $|\alpha_{\mathrm{sp}(\Gamma_j)}|>a^*|\Gamma_j|$ or else the reverse inequality holds. When the former occurs, we use (4.10) and write

$$\int_{\theta(q_x)=\theta_{\Gamma_j}(x),x\in \mathrm{sp}(\Gamma_j)} \nu_{\mathrm{sp}(\Gamma_j)}(dq)\ e^{-H_\lambda(q|q',p,p')}\mathbf{1}_{|\alpha_{\mathrm{sp}(\Gamma_j)}|>a^*|\Gamma_j|}$$
$$\le e^{-C_P|\Gamma_j|+1} Z^{(\{\epsilon_i\}_j,q',p,p')}_{\lambda,\mathrm{sp}(\Gamma_j)}\Big(|\alpha_{\mathrm{sp}(\Gamma_j)}|\le a^*|\Gamma_j|\Big)$$

where $\{\epsilon_i\}_j$ are the values of θ on the boundaries of $\mathrm{sp}(\Gamma_j)$, which are fixed by $\underline{\Gamma}$.

Collecting the above inequalities and calling $|\Gamma_{I^c}|$, $I\sqsubset\{0,..,n\}$, the sum of $|\Gamma_j|$ over $j\in\{0,..,n\}\setminus I$, we get from (4.26)

$$Z^{(+,p,p')}_{\lambda,\Lambda^-}\Big(q\to\underline{\Gamma}\Big)\le \sum_{I\sqsubset\{0,..,n\}} e^{-C_P|\Gamma_{I^c}|+n+1}$$
$$\times Z^{(+,p,p')}_{\lambda,\Lambda^-}\Big(\Big\{\Big|\tfrac{\alpha_{\Delta^-}}{|\Delta^-|}-\alpha_-\Big|\le 2\zeta\Big\},\Big\{\Big|\tfrac{\alpha_{\mathrm{sp}\Gamma_i}}{|\Gamma_i|}\Big|\le a^*,\forall i\Big\},\Big\{q\to\Gamma_i,i\in I\Big\}\Big)$$

having used that $\{\theta=-1 \text{ on } \Delta^-\setminus\Delta^-_0\}$ implies that

$$\Big|\alpha_{\Delta^-\setminus\Delta^-_0}-\alpha_-|\Delta^-\setminus\Delta^-_0|\Big|\le\zeta|\Delta^-\setminus\Delta^-_0|$$

We next change λ into $\lambda=0$ by exploiting the constraints in the last partition function:

$$Z^{(+,p,p')}_{\lambda,\Lambda^-}\Big(q\to\underline{\Gamma}\Big)\le \sum_{I\sqsubset\{0,..,n\}} e^{-C_P|\Gamma_{I^c}|+\lambda(\beta\ell^d)(\alpha_-+2\zeta)|\Lambda^-|} \tag{4.27}$$
$$\times e^{n+1+\lambda(\beta\ell^d)[-(\alpha_-+2\zeta)+a^*]|\underline{\Gamma}|} Z^{(+,p,p')}_{0,\Lambda^-}\Big(\{q\to\Gamma_i,i\in I\}\Big)$$

and use the Peierls bound

$$Z^{(+,p,p')}_{0,\Lambda^-}\Big(\{q\to\Gamma_i,i\in I\}\Big)\le e^{-C_P|\Gamma_I|}Z^{(+,p,p')}_{0,\Lambda^-}$$

We will next reconstruct a partition function with the correct value of λ and then conclude the proof of the theorem. By (4.15),

$$Z^{(+,p,p')}_{0,\Lambda^-}\le(1-ce^{-C_P/4})^{-1}Z^{(+,p,p')}_{0,\Lambda^-}\Big(|\alpha_{\Lambda^-}-\alpha_+|\Lambda^-||\le 2\zeta|\Lambda^-|\Big)$$

and recalling C_P is so large that $(1-ce^{-C_P/4})^{-1}\le e$,

$$Z^{(+,p,p')}_{0,\Lambda^-}\le e^{1-\lambda(\beta\ell^d)(\alpha_+-2\zeta)|\Lambda^-|}Z^{(+,p,p')}_{\lambda,\Lambda^-}$$

By the two inequalities above, (4.27) becomes

$$Z^{(+,p,p')}_{\lambda,\Lambda^-}\Big(q\to\underline{\Gamma}\Big) \le 2^{n+1}e^{-C_P|\underline{\Gamma}|-\lambda(\beta\ell^d)(\alpha_+-\alpha_--4\zeta)|\Lambda^-|}$$
$$\times e^{n+2+\lambda(\beta\ell^d)[-(\alpha_-+2\zeta)+a^*]|\underline{\Gamma}|}Z^{(+,p,p')}_{\lambda,\Lambda^-}$$

with 2^{n+1} counting the cardinality of the subsets I in (4.27). By the choice of ζ, see (2.5),

$$\alpha_+-\alpha_--4\zeta=\frac{\alpha_+-\alpha_-}{2}+\frac{\alpha_+-\alpha_-}{10}$$

and by choosing properly the constant $C(d)$ in Theorem 2.1,

$$\{-\frac{7C_P}{8}+\delta^{-1}C_P\lambda[-\alpha_-+a^*]\}|\underline{\Gamma}|+(n+2)+(n+1)\log 2\le-\frac{C_P}{4}|\underline{\Gamma}| \tag{4.28}$$

we finally get:

$$\frac{Z^{(+,p)}_{\lambda,V(\Gamma_0)}\Big(|\Lambda^-|\ge V\Big)}{Z^{(+,p)}_{\lambda,V(\Gamma_0)}} \tag{4.29}$$
$$\le e^{-C_P|\Gamma_0|/8-\omega V}\sum_{\underline{\Gamma}:|\Lambda^-|\ge V} e^{-C_P|\underline{\Gamma}|/4-\lambda(\beta\ell^d)(\alpha_+-\alpha_-)|\Lambda^-|/10}$$

We will prove that

$$\hat m_\lambda(\Gamma_0;p):=\sum_{\underline{\Gamma}} e^{-C_P|\underline{\Gamma}|/4-\lambda(\beta\ell^d)(\alpha_+-\alpha_-)|\Lambda^-|/10}\le 1 \tag{4.30}$$

which will conclude the proof of the theorem.

A counting argument

The proof of (4.30) will use coarse graining with grains the cubes of size $\xi_+(\lambda)$ of a partition $\mathcal{D}^{(\xi_+(\lambda))}$ of $\mathbb{Z}^d$, where

$$\xi_+(\lambda)=\xi_-(\lambda)^{10} \tag{4.31}$$

supposing for simplicity $\xi_+(\lambda)\in\mathbb{N}$.

Call $\{\hat C_x, x\in X\}$, X a label set, the cubes of $\mathcal{D}^{(\xi_+(\lambda))}$ with non empty intersection with $V(\Gamma_0)$, in the sequel we will restrict to such a set. Given $\underline{\Gamma}$, each $\hat C_x, x\in X$, falls into one of the following three categories:

- $\hat C_x\sqsubset\Delta^-(\underline{\Gamma})$, • $\hat C_x\sqcap \mathrm{sp}(\underline{\Gamma})\ne\emptyset$, • $\hat C_x\sqsubset \mathrm{int}^+(\underline{\Gamma})$

In fact the fourth possibility, namely $\hat{C}_x \setminus \Delta^-(\underline{\Gamma}) \neq \emptyset$, $\hat{C}_x \setminus \mathrm{int}^+(\underline{\Gamma}) \neq \emptyset$ and $\hat{C}_x \sqcap \mathrm{sp}(\underline{\Gamma}) = \emptyset$, is empty because of the definition of contours.

To distinguish the three cases we introduce the variable $\psi(\underline{\Gamma}; x)$, $x \in X$, which has value -1, 0 and 1 respectively in the first, second and third case above. For ease of reference we write it down explicitly:

$$\psi(\underline{\Gamma}; x) = \begin{cases} -1 & \text{if } \hat{C}_x \sqsubset \Delta^-(\underline{\Gamma}) \\ 0 & \text{if } \hat{C}_x \sqcap \mathrm{sp}(\underline{\Gamma}) \neq \emptyset \\ 1 & \text{if } \hat{C}_x \sqsubset \mathrm{int}^+(\underline{\Gamma}) \end{cases} \tag{4.32}$$

For each $x \in X$ we introduce a weight $w_\lambda(\underline{\Gamma}; x)$ which is determined by the value of $\psi(\underline{\Gamma}; x)$. We first call, for $x \in X$,

$$\hat{C}_x^+ = \big\{ y : \mathrm{dist}(y, \hat{C}_x) \leq \xi_+(\lambda) \big\}, \qquad \underline{\Gamma}_x = \{\Gamma_i \in \underline{\Gamma} : \mathrm{sp}(\Gamma_i) \sqcap \hat{C}_x \neq \emptyset\} \tag{4.33}$$

and then define the weights

$$w_\lambda(\underline{\Gamma}; x) = \begin{cases} e^{-\lambda(\beta\ell^d)[(\alpha_+ - \alpha_-)/10]\; \xi_+(\lambda)^d} & \text{if } \psi(\underline{\Gamma}; x) = -1 \\ e^{-3^{-d}(C_P/4)|\mathrm{sp}(\underline{\Gamma}_x) \sqcap C_x^+|} & \text{if } \psi(\underline{\Gamma}; x) = 0 \\ 1 & \text{if } \psi(\underline{\Gamma}; x) = 1 \end{cases} \tag{4.34}$$

The definition is such that

$$\hat{m}_\lambda(\Gamma_0; p) \leq \sum_{\underline{\Gamma}} \prod_{x \in X} w_\lambda(\underline{\Gamma}; x) \tag{4.35}$$

The difficulty with (4.35) is that the weights $w_\lambda(\cdot; x)$ for different x are not independent, but, as we will see, we can reduce by upper bounds to the independent case. Let

$$S = \Big\{ s \in \{0, \pm 1\}^X : s_x = 0 \text{ for } \hat{C}_x \sqcap \mathrm{sp}(\Gamma_0) \neq \emptyset, \bigsqcup_{s_x < 1} \hat{C}_x \text{ is } \star \text{ connected} \Big\} \tag{4.36}$$

Our goal is to define new weights $\omega_\lambda(s_x)$, $x \in X$, so that

$$\hat{m}_\lambda(\Gamma_0; p) \leq \sum_{s \in S} \prod_{x \in X} \omega_\lambda(s_x) \tag{4.37}$$

Recalling (4.34) we set $\omega_\lambda(1) = 1$ and

$$\omega_\lambda(-1) = e^{-\lambda(\beta\ell^d)[(\alpha_+ - \alpha_-)/10]\; \xi_+(\lambda)^d} \tag{4.38}$$

To define $\omega_\lambda(0)$, we denote by $\underline{T} = (T_1, .., T_n)$ any finite sequence of disjoint, $\star$ connected, $\mathcal{D}^{(1)}$-measurable sets and define, given a cube $\hat{C} \in \mathcal{D}^{(\xi_+(\lambda))}$,

$$\omega_\lambda(0) = \sum_{\substack{\underline{T}: T_i \sqcap \hat{C} \neq \emptyset \\ |T_i| \geq N^*}} \prod_i 3^{|T_i|} e^{-3^{-d}(C_P/4)|T_i|} \tag{4.39}$$

where

$$N^* = \min\Big\{ c_d \xi_-(\lambda)^{d-1} \ ; \ \xi_+(\lambda) \Big\} \tag{4.40}$$

c_d an isoperimetric constant.

We will prove (4.37) by showing that

$$\sum_{\underline{\Gamma}: \psi(\underline{\Gamma}; x) = s_x, x \in X} \ \prod_{x \in X} w_\lambda(\underline{\Gamma}; x) \ \leq \ \prod_{x \in X} \omega_\lambda(s_x) \tag{4.41}$$

Indeed the factors $3^{|T_i|}$ in (4.39) take into account that there at most $3^{|T_i|}$ contours Γ with $\mathrm{sp}(\Gamma) = T_i$. N^* is a lower bound for the number of sites of all sets $\mathrm{sp}(\Gamma)$ which are in $\hat{C}_x^+$, over all fat contours Γ such that $\mathrm{sp}(\Gamma) \sqcap \hat{C}_x \neq \emptyset$: the first term is a lower bound for $|\Gamma|$ and covers the case when $\mathrm{sp}(\Gamma) \sqsubset \hat{C}_x^+$ (see (4.7) and (4.3)), the second one is for the case when $\mathrm{sp}(\Gamma)$ is not contained in $\hat{C}_x^+$. With this choice (4.41) holds.

We next need to bound the weights $\omega(\cdot)$. For $\hat{\lambda}$ small enough N^* becomes large too, and if C_P is also large enough,

$$\begin{aligned} \omega_\lambda(0) \quad &\leq \quad e^{-3^{-d}(C_P/8)N^*} \Big(1 + \sum_{T \ni 0, |T| \geq N^*} 3^{|T|} e^{-3^{-d}(C_P/8)|T|} \Big)^{[\xi_+(\lambda)]^d} \\ &\leq \quad 2\, e^{-3^{-d}(C_P/8)N^*} \end{aligned} \tag{4.42}$$

where the last estimate is obtained in the same way as the bound 3.5.

Recalling (4.34),(4.31) and (4.2)

$$\omega_\lambda(-1) = e^{-\lambda(\beta \ell^d)[(\alpha_+ - \alpha_-)/10]\ \xi_+(\lambda)^d}, \qquad \lambda \xi_+(\lambda)^d = \lambda^{1-5d} \tag{4.43}$$

Thus by choosing λ^* small enough, also $\omega_\lambda(-1)$ can be made as small as desired, and consequently we can achieve that $\hat{m}_\lambda(\Gamma_0; p) < 1$ and (4.1) is proved.

5. Uniqueness of the plus and minus phases

In this section we will prove that when $0 < \lambda \le \lambda^*$ any tempered DLR measure μ is in $\mathcal{G}^+_\lambda$, the proof of the analogous statement for $\lambda < 0$ is omitted. The idea of the proof is similar to that in [**8**] and [**1**].

Let Λ_0 be a cube of side L and Λ' the cube with same center and side $L/2$. Given a configuration q, let $G = G(q)$ be the union of the maximal connected components of $\{x : \Theta(q; x) < 1\}$ which have intersection with Λ_0^c. Call

$$G^+ = \{x : \mathrm{dist}(x, G) \le 1\}, \quad \Delta = (G^+)^c \sqcap \Lambda_0 \tag{5.1}$$

Given any bounded, local function f, we consider below L so large that $f = f(q_{\Lambda'})$ (i.e. it only depends on $q_{\Lambda'}$). We then have

$$\int \mu(dq) f(q) = \sum_{\Delta \not\sqsupset \Lambda'} \int \mu(dq) f(q) \mathbf{1}_{\Delta(q)=\Delta} + \sum_{\Delta \sqsupset \Lambda'} \int \mu(dp) \int \mu^{(+,p)}_{\lambda,\Delta}(dq_{\Lambda'}) f(q_{\Lambda'}) \mathbf{1}_{\Delta(p)=\Delta}$$

Recalling that the dependence of Λ, Δ, Λ' on L is not made explicit, we will prove

$$\lim_{L\to\infty} \mu\big(\Delta \not\sqsupset \Lambda'\big) = 0 \tag{5.2}$$

thus concluding that

$$\int \mu(dq) f(q) = \lim_{L\to\infty} \sum_{\Delta \sqsupset \Lambda'} \int m_L(dp; \Delta) \int \mu^{(+,p)}_{\lambda,\Delta}(dq_{\Lambda'}) f(q_{\Lambda'}) \tag{5.3}$$

$$m_L(dp; \Delta) = \Big(\frac{\mathbf{1}_{\Delta(p)=\Delta}}{\mu\big(\Delta \sqsupset \Lambda'\big)}\Big) \mu(dp), \qquad \sum_{\Delta \sqsupset \Lambda'} \int m_L(dp; \Delta) = 1$$

For any L the r.h.s. of (5.3) is the expectation of f w.r.t. a measure in $\mathcal{G}^+_{\lambda,\Lambda'}$, hence, by the arbitrariness of f, $\mu \in \mathcal{G}^+_\lambda$.

In the remaining part of the section we will prove (5.2).

The random sets G^-_k.

It is convenient, at this point, to use the function $\theta(q_x)$ rather than the phase indicator $\Theta(q; x)$, due to the local structure of the former. Recalling (2.2),

$$\{\Theta < 1\} = \{x : \mathrm{dist}(x, \{\theta < 1\}) \le 1\}$$

so that the maximal connected components of $\{\Theta < 1\}$ are obtained from the maximal 3-connected components of $\{\theta < 1\}$ (two sites x and y being called 3-connected if $|x-y| \le 3$) by taking all sites which have distance ≤ 1 from the latter. As a consequence, defining $G^- = G^-(q) = \{x \in G : \theta(q_x) < 1\}$, we recover G from G^- by setting $G = \{x : \mathrm{dist}(x, G^-) \le 1\}$; moreover, the part of G^- contained in a maximal connected component of G is a 3-connected set and distinct 3-connected components of G^- belong to distinct connected components of G. G^- can be defined directly as the union of all the maximal 3-connected components of $\{x : \theta(q_x) < 1\}$ which have distance ≤ 1 from Λ_0^c.

We will next approximate the set G^- by sets G_k^-. For $k \ge 1$, let

$$\Lambda_k = \Lambda_{k-1} \setminus \{x : \mathrm{dist}(x, \Lambda_{k-1}^c) \le 3\} \tag{5.4}$$

and define G_k^- as equal to the union of all the maximal 3-connected components of $\{x \in \Lambda_k^c : \theta(q_x) < 1\}$ which have distance ≤ 1 from Λ_0^c. We call $M_k^- = G_k^- \sqcap (\Lambda_{k-1} \setminus \Lambda_k)$. Since $M_k^- = \emptyset$ implies that $\theta(q_x) = 1$ for all $1 \le \mathrm{dist}(x, G_k^-) \le 3$, it then follows that $G^- = G_k^-$. Thus $G^- \sqsubset \Lambda_{k-1}^c$ and consequently $G \sqsubset \Lambda_k^c$, so that

$$\text{(5.2) is implied by} \quad \lim_{L\to\infty} \mu\Big(\{\text{there is } k\text{: } M_k^- = \emptyset, \Lambda_k \sqsupset \Lambda'\}\Big) = 1 \tag{5.5}$$

The random times s_j.

We will prove (5.5) by successive approximations, first showing that $|M_k^-|$ cannot be too large, too often. Call

$$a(j) = d - 1 - j + \frac{j}{2d}, \qquad j = 1, .., d \tag{5.6}$$

and, setting $s_0 = 0$, define the "random times" s_j, $j = 1, .., d$, as

$$s_j = \begin{cases} \min\left\{k : |M_k^-| \le L^{a(j)}\right\}, & \text{if the set is non empty} \\ +\infty & \text{otherwise} \end{cases} \tag{5.7}$$

Since $a(j)$ is decreasing, $s_j \ge s_{j-1}$, $j = 1, .., d$. Moreover we can read off from $G_k^-(q)$ which are the values of s_j which are not larger than k, if any. Thus there exists a family of sets $K_{j,k}^0$ so that

$$\{s_j = k\} \quad \Leftrightarrow \quad \{G_k^- \in K_{j,k}^0\} \tag{5.8}$$

For L sufficiently large, if $s_d < \infty$ then $s_d = \min\{k \geq s_{d-1} : M_k^- = \emptyset\}$. Thus, by choosing a suitably small constant b, we conclude that in the set $\{s_{j+1} - s_j \leq bL, j < d\}$, $\Lambda_{s_d} \sqsupset \Lambda'$ and, for L large enough, $M_{s_d}^- = \emptyset$. Recalling (5.5), the statement we want to prove, namely that $\mu \in \mathcal{G}_\lambda^+$, then follows from

$$\lim_{L\to\infty} \mu\Big(\{s_{j+1}-s_j > bL\}\sqcap\{s_j \leq jbL\}\Big) = 0, \qquad j = 0,..,d-1 \tag{5.9}$$

We shorthand

$$\{|q|_\Delta \leq L^\delta\} = \{q : |q_x| \leq L^\delta, \text{ for all } x \in \Delta\}, \qquad \delta > 0,\ \Delta \sqsubset \mathbb{Z}^d \tag{5.10}$$

Since μ is tempered, for any $\delta > 0$,

$$\lim_{L\to\infty} \mu\big(\{|q|_{\Lambda_0} \leq L^\delta\}\big) = 1 \tag{5.11}$$

In fact

$$\mu\big(\{|q|_{\Lambda_0} > L^\delta\}\big) \leq L^d \sup_{x\in\Lambda_0} \mu\big(|q_x| > L^\delta\big)\big)$$

and then (5.11) follows from (2.12), with n large enough.

By (5.11), we can replace the condition (5.9) by

$$\lim_{L\to\infty} \mu\Big(\{s_{j+1} - s_j > bL\} \sqcap \{s_j \leq jbL\} \sqcap \{|q|_{\Lambda_0} \leq L^\delta\}\Big) = 0, \quad j = 0,..,d-1 \tag{5.12}$$

or, for the generic $j \in \{0,..,d-1\}$ and with $K_{j,k}^0$ defined in (5.8),

$$\lim_{L\to\infty} \sum_{k\leq jbL} \sum_{G_k^-\in K_{j,k}^0} \mu\Big(\{s_{j+1}-s_j > bL\}\sqcap\{G_k^-(q) = G_k^-\}\sqcap\{|q|_{\Lambda_0} \leq L^\delta\}\Big) = 0 \tag{5.13}$$

The set $\mathcal{B}_k$.

We will next suitably modify the set $\{s_{j+1} - s_j > bL\}$. Let $\mathcal{B}_k$ be the union of all the maximal connected components of

$$\{x : \theta(q_x) < 1\} \sqcap \{x : \mathrm{dist}(x, G_k^-) > 3)\}$$

which have non empty intersection with $\{y : \mathrm{dist}(y, M_k^-) \leq 6\}$.

The set G_h^-, $h > k$, contains G_k^-, and the sites $x \in G_h^- \setminus G_k^-$, $\mathrm{dist}(x, G_k^-) \leq 3$, have distance ≤ 3 from M_k^-; hence

$$\{x \in G_h^- \setminus G_k^- : \mathrm{dist}(x, M_k^-) > 3\} \sqsubset \mathcal{B}_k \tag{5.14}$$

Since

$$G_h^- \setminus G_k^- \sqsupset \bigsqcup_{k\le h'\le h} M_{h'}$$

then, in the set $\{q : s_{j+1}(q) - s_j(q) > bL;\ G_k^-(q) = G_k^-\}$, $G_k^- \in K_{j,k}^0$,

$$\left|G_h^- \setminus G_k^-\right| \ge bL^{1+a(j+1)}, \qquad h \ge k + bL$$

and by (5.14)

$$|\mathcal{B}_k| \ge bL^{1+a(j+1)} - 50^d |M_k^-|$$

On the other hand $|M_k^-| \le L^{a(j)}$ and $a(j+1)+1-a(j) = 1/(2d) > 0$, so that, for L large enough,

$$|\mathcal{B}_k| \ge \frac{b}{2} L^{1+a(j+1)} \tag{5.15}$$

Thus, instead of (5.13) we may prove as well that, for any $j \in \{0, .., d-1\}$,

$$\lim_{L\to\infty} \sum_{k\le jbL} \sum_{G_k^- \in K_{j,k}^0} \mu\Big(\big\{G_k^-(q)=G_k^-\big\} \sqcap \big\{|q|_{\Lambda_0} \le L^\delta\big\} \sqcap \big\{|\mathcal{B}_k| \ge \tfrac{b}{2} L^{1+a(j+1)}\big\}\Big) = 0 \tag{5.16}$$

Let $G_k^- \in K_{j,k}^0$ in the sequel and call

$$\Delta_k = \{x \in \Lambda_0 : \mathrm{dist}(x, G_k^-) \ge 3\}, \quad \partial\Delta_k = \{x \in \Delta_k : \mathrm{dist}(x, G_k^-) = 3\} \tag{5.17}$$

When $G_k^-(q) = G_k^-$, then $\theta(q_x) = 1$ for all $x \in \partial\Delta_k \sqcap \Lambda_k^c$; thus

$$\{G_k^-(q) = G_k^-\} = K_{j,k} \sqcap \{\theta(q_x) = 1, x \in \partial\Delta_k \sqcap \Lambda_k^c\} \tag{5.18}$$

where

$$K_{j,k} = \big\{\theta(q_x) < 1, x \in G_k^-\big\} \sqcap \big\{\theta(q_x) = 1, x \in \Lambda_k^c, \mathrm{dist}(x, G_k^-) \in \{1,2\}\big\} \tag{5.19}$$

so that $K_{j,k} \in \mathcal{F}_{\Delta_k^c \sqcap \Lambda_k^c}$. We then have

$$\begin{aligned} &\mu\Big(\{G_k^-(q) = G_k^-\} \sqcap \{|q|_{\Lambda_0} \le L^\delta\} \sqcap \{|\mathcal{B}_k| \ge \frac{b}{2} L^{1+a(j+1)}\}\Big) \qquad (5.20)\\ &\qquad = \int_{K_{j,k} \sqcap \{|q'|_{\Lambda_0 \setminus \Delta_k} \le L^\delta\}} \mu(dq')\ \mu_{\lambda,\Delta_k}^{(q')}\Big(\{\theta(q_x) = 1, x \in \partial\Delta_k \sqcap\ \Lambda_k^c\}\\ &\qquad\qquad \sqcap \{|q|_{\Delta_k} \le L^\delta\} \sqcap\ \{|\mathcal{B}_k| \ge \frac{b}{2} L^{1+a(j+1)}\}\Big) \end{aligned}$$

To bound the last term on the r.h.s. of (5.20), we write it as a fraction of two partition functions:

$$\mu^{(q')}_{\lambda,\Delta_k}\Big(\theta = 1 \text{ on } \partial\Delta_k \sqcap \Lambda_k^c;\ |q|_{\Delta_k} \le L^\delta;\ |\mathcal{B}_k| \ge \frac{b}{2} L^{1+a(j+1)}\Big) = \frac{N^{(q')}_{\lambda,\Delta_k}}{Z^{(q')}_{\lambda,\Delta_k}} \tag{5.21}$$

where

$$N^{(q')}_{\lambda,\Delta_k} = Z^{(q')}_{\lambda,\Delta_k}\Big(\theta = 1 \text{ on } \partial\Delta_k \sqcap \Lambda_k^c;\ |q|_{\Delta_k} \le L^\delta;\ |\mathcal{B}_k| \ge \frac{b}{2} L^{1+a(j+1)}\Big)$$

Reduction to + boundary conditions.

By using (2.15) we will replace $N^{(q')}_{\lambda,\Delta_k}$ by a new partition function with + boundary conditions. Calling boundary sites, the sites $x \in \Delta_k^c : \text{dist}(x,\Delta_k) \le 2$, we want to change q' at all the boundary sites where $\theta < 1$ and also to impose that $\theta = 1$ on the whole $\partial\Delta_k$ and not only on $\partial\Delta_k \sqcap \Lambda_k^c$, as in the constraint of the partition function defining $N^{(q')}_{\lambda,\Delta_k}$. The whole thing works because the overall number of sites in this operation is "small", as we are going to see.

The boundary sites have distance equal to 1 or to 2 from G_k^-, hence, by the definition of G_k^-, $\theta = 1$ on all such sites which are not in Λ_k. Thus the new b.c. q'^+ is defined in such a way that $q'^+_x = q'_x$ for all $x \notin A_k$, $A_k = \{y \in \Delta_k^c \sqcap \Lambda_k : \text{dist}(y,\Delta_k) \le 2\}$; while $\theta(q'^+_x) = 1$ for all $x \in A_k$. Since $|\partial\Delta_k \sqcap \Lambda_k| \le 50^d |M_k^-|$, by (2.15),

$$N^{(q')}_{\lambda,\Delta_k} \le e^{c(\beta\ell^d)50^d|M_k^-|L^{\delta n}} Z^{(q'+)}_{\lambda,\Delta_k}\Big(\theta = 1 \text{ on } \partial\Delta_k;\ |\mathcal{B}_k| \ge \tfrac{b}{2} L^{1+a(j+1)}\Big) \tag{5.22}$$

where on the r.h.s. we have dropped the condition $|q|_{\Delta_k} \le L^\delta$. The r.h.s. is a + partition function in the sense of (2.8) and we have

$$\frac{Z^{(q'+)}_{\lambda,\Delta_k}\Big(\theta = 1 \text{ on } \partial\Delta_k;\ |\mathcal{B}_k| \ge \tfrac{b}{2} L^{1+a(j+1)}\Big)}{Z^{(q'+)}_{\lambda,\Delta_k}\Big(\theta = 1 \text{ on } \partial\Delta_k\Big)} = \mu^{(+,q'+)}_{\lambda,\Delta_k}\Big(|\mathcal{B}_k| \ge \tfrac{b}{2} L^{1+a(j+1)}\Big) \tag{5.23}$$

The r.h.s. will give the small factor which makes the sum in (5.16) convergent and infinitesimal with L, but let us first readjust the partition function in the denominator of the fraction on the l.h.s. of

(5.23):

$$\frac{Z^{(q'+)}_{\lambda,\Delta_k}\big(\theta=1 \text{ on } \partial\Delta_k\big)}{Z^{(q'+)}_{\lambda,\Delta_k}\big(\theta=1 \text{ on } \partial\Delta_k,\, |q|_{\Delta_k}\le L^\delta\big)} \le \Big(1 - L^d \sup_{x\in\Delta_k} \mu^{(+,q'+)}_{\lambda,\Delta_k}\big(|q_x| > L^\delta\big)\Big)^{-1}$$

By (2.12), for L large enough, the last factor is smaller than 2 and, using again (2.15), we get

$$Z^{(q'+)}_{\lambda,\Delta_k}\big(\theta=1 \text{ on } \partial\Delta_k\big) \le 2e^{c(\beta\ell^d)50^d|M_k^-|L^{\delta n}} Z^{(q')}_{\lambda,\Delta_k}\big(\theta=1 \text{ on } \partial\Delta_k,\, |q|_{\Delta_k}\le L^\delta\big) \quad (5.24)$$

hence, going back to (5.20) and with c' a suitable constant,

$$\begin{aligned}
&\mu\Big(\{G_k^-(q)=G_k^-\}\sqcap\{|q|_{\Lambda_0}\le L^\delta\}\sqcap\{|\mathcal{B}_k|\ge \frac{b}{2}L^{1+a(j+1)}\}\Big)\\
&\quad\le e^{c'(\beta\ell^d)|M_k^-|L^{\delta n}} \int_{K_{j,k}\sqcap\{|q|_{\Lambda_0\setminus\Delta_k}\le L^\delta\}} \mu(dq')\ \mu^{(q')}_{\lambda,\Delta_k}\big(\theta=1 \text{ on } \partial\Delta_k\big)\\
&\qquad\qquad\times\mu^{(+,q'+)}_{\lambda,\Delta_k}\big(|\mathcal{B}_k|\ge \frac{b}{2}L^{1+a(j+1)}\big) \qquad (5.25)
\end{aligned}$$

The probability that $|\mathcal{B}_k| \ge \frac{b}{2}L^{1+a(j+1)}$

By the definition of $\mathcal{B}_k$, a maximal connected component of $\mathcal{B}_k$ is the layer attached to a plus contour Γ whose spatial support has non empty intersection with $F := \{y : \mathrm{dist}(y, M_k^-) \le 6\}$. Let $\underline{\Gamma} = (\Gamma_1, .., \Gamma_n)$ be the collection of all such plus contours, then, by Theorem 4.5,

$$\begin{aligned}
&\mu^{(+,q'+)}_{\lambda,\Delta_k}\big(|\mathcal{B}_k|\ge \frac{b}{2}L^{1+a(j+1)}\big)\\
&\quad\le \sum_{\substack{\underline{\Gamma}=(\Gamma_1,..,\Gamma_n)\\ \mathrm{sp}(\Gamma_i)\sqcap F\ne\emptyset, 1\le i\le n}} e^{-\beta(C_p/8)\,|\underline{\Gamma}|} \sum_{\substack{V_i, 1\le i\le n\\ \sum_i V_i\ge \frac{b}{2}L^{1+a(j+1)}}} \prod_i e^{-\omega V_i}\\
&\quad\le e^{-\omega(b/4)L^{1+a(j+1)}} \sum_{\substack{\underline{\Gamma}=(\Gamma_1,..,\Gamma_n)\\ \mathrm{sp}(\Gamma_i)\sqcap F\ne\emptyset, 1\le i\le n}} e^{-\beta(C_p/8)\,|\underline{\Gamma}|}[1-e^{-\omega/2}]^{-n}
\end{aligned}$$

where we have used (4.23) for the fat contours in $\underline{\Gamma}$ and (4.4) for the slim ones. The number n of contours cannot exceed $|F|$ so that the above is bounded by

$$e^{-\omega(b/4)L^{1+a(j+1)}}[1-e^{-\omega/2}]^{-|F|}\prod_{x\in F}\Big(1+\sum_{\Gamma:\mathrm{sp}(\Gamma)\ni x} e^{-(C_p/8)\,|\Gamma|}\Big)$$

For C_P as large as required by Lemma 3.1,

$$\sum_{\Gamma:\mathrm{sp}(\Gamma)\ni x} e^{-\beta(C_p/8)\,|\Gamma|}\le \kappa e^{-C_P/8}$$

and since $|F|\le c|M_k|\le c'L^{a(j)}$, we get

$$\begin{aligned}&\mu^{(+,q'+)}_{\lambda,\Delta_k}\big(|\mathcal{B}_k|\ge \frac{b}{2}L^{1+a(j+1)}\big)\\ &\qquad\le \exp\{-\omega(b/4)L^{1+a(j+1)}+c''L^{a(j)}\}\le \exp\{-\omega(b/8)L^{1+a(j+1)}\}\end{aligned}$$

for L large enough, because

$$a(j)=d-1-j+\frac{j}{2d},\quad 1+a(j+1)=d-1-j+\frac{j+1}{2d}$$

Conclusions.

To prove (5.13), we write

$$\begin{aligned}&\text{l.h.s. of (5.13)}\ \le\\ &\quad\le \lim_{L\to\infty}\sum_{k\le jbL}\sum_{G_k^-\in K^0_{j,k}}\exp\left\{c'(\beta\ell^d)|M_k^-|L^{\delta n}-\omega(b/8)L^{1+a(j+1)}\right\}\\ &\qquad\qquad\times\int_{K_{j,k}\sqcap|q|_{\Lambda_0\setminus\Delta_k}\le L^\delta}\mu(dq')\ \mu^{(q')}_{\lambda,\Delta_k}\big(\theta=1\text{ on }\partial\Delta_k\big)\end{aligned}$$

By (5.7), $|M_k^-|\le L^{a(j)}$, and with such a bound, the first factor on the r.h.s. becomes independent of k. Since

$$\int_{K_{j,k}\sqcap|q|_{\Lambda_0\setminus\Delta_k}\le L^\delta}\mu(dq')\ \mu^{(q')}_{\lambda,\Delta_k}\big(\theta=1\text{ on }\partial\Delta_k\big)\le\mu\Big(\{G_k^-(q)=G_k^-\}\Big)$$

we have a sum of probabilities of disjoint events, hence

$$\text{l.h.s. of (5.13)}\quad\le\quad\lim_{L\to\infty}e^{c'L^{a(j)}(\beta\ell^d)L^{n\delta}-\omega(b/8)L^{1+a(j+1)}}=0$$

if δ is chosen small enough.

6. The LMP model

As an example of the possible applications of Theorem 2.1, we consider the LMP model for phase transitions in the continuum, [**9**]. Analogous considerations apply to Ising systems with Kac potentials, as in [**2**] and [**4**].

The LMP model

As already mentioned in Section 2, the variable q_x in the LMP model is a particle configuration in the cell C_x, C_x being a cube of a partition $\mathcal{D}^{(\ell_+,\gamma)}$ of $\mathbb{R}^d$ which will be described later. $|q_x|$ denotes the number of particles in q_x and $|q|$ the total number of particles in the configuration q.

The LMP energy of a finite particle configuration $q = (..., r_i, ..)$ is

$$H_{\gamma,\lambda,\beta}(q) = \int_{\mathbb{R}^d} \beta e_\lambda(j_\gamma * q) dr \tag{6.1}$$

where, $\gamma > 0$ is the Kac scaling parameter,

$$e_\lambda(x) = -\lambda x - \frac{x^2}{2} + \frac{x^4}{4!} \tag{6.2}$$

$$j_\gamma * q(r) = \sum_{r_i \in q} j_\gamma(r, r_i) \tag{6.3}$$

with

$$j_\gamma(r, r') := \gamma^d j(\gamma r, \gamma r') \tag{6.4}$$

and $j(r, r')$ a translational invariant, smooth, symmetric, probability kernel supported by $|r - r'| \le 1$.

In the language of Section 2,

$$\alpha_x(q) = \frac{|q \sqcap C_x|}{\beta \ell^d}, \qquad \ell^d = |C_x|$$

while the chemical potential λ above differs from λ in Section 2 by an additive constant, $\lambda \to \lambda - \lambda_{\beta,\gamma}$, see the paragraph "Peierls estimates" below.

Mean field

The mean field limit, formally corresponding to $\gamma = 0$, is described by the free energy density

$$f_{\lambda,\beta}(x) = e_\lambda(x) - \frac{s(x)}{\beta} \tag{6.5}$$

where $s(\cdot)$ is the entropy density

$$s(x) = -x(\log x - 1) \tag{6.6}$$

For $\beta < \beta_c = (3/2)^{3/2}$, $f''_{\lambda,\beta}(x) > 0$ for all λ and x. For any $\beta > \beta_c$ there is a unique value of λ, $\lambda = \lambda_\beta$, where $f_{\lambda_\beta,\beta}$ has two distinct minimizers, $\rho_\beta^\pm$, elsewhere the minimizer is unique. Minimizers are solutions of the mean field equation $f'_{\lambda,\beta}(x) = 0$ which can be written as

$$x = K_\beta(x) := e^{-\beta e'_{\lambda_\beta}(x)} \tag{6.7}$$

It has three solutions: $0 < \rho_\beta^- < \rho_\beta^0 < \rho_\beta^+$, and

$$f''_{\lambda_\beta,\beta}(\rho_\beta^\pm) > 0 \tag{6.8}$$

There is $\beta_0 > \beta_c$ so that

$$K'_\beta(\rho_\beta^+) \in (-1,1) \quad \text{for all } \beta \in (\beta_c, \beta_0)$$

while $K'_\beta(\rho_\beta^-) \in (0,1)$ for all $\beta > \beta_c$. Following LMP we restrict to $\beta \in (\beta_c, \beta_0)$.

Contours

Contours are defined using two scales,

$$\ell_{\pm,\gamma} = \gamma^{-(1\pm\epsilon)}$$

where $\epsilon > 0$ is a "very small" parameter. We then denote by $\mathcal{D}^{(\ell_{\pm,\gamma})}$ two partitions of $\mathbb{R}^d$ into cubes of side $\ell_{\pm,\gamma}$, supposing $\mathcal{D}^{(1)}$ finer than $\mathcal{D}^{(\ell_{-,\gamma})}$ finer than $\mathcal{D}^{(\ell_{+,\gamma})}$. Thus the cell length ℓ of the previous sections is $\ell = \ell_{+,\gamma} = \gamma^{-(1+\epsilon)}$ in the present model.

We then define

$$\eta^{(\zeta,\ell_{-,\gamma})}(q;r) = \begin{cases} \pm 1 & \text{if } \left| \ell_{-,\gamma}^{-d} |q \sqcap C_r^{(\ell_{-,\gamma})}| - \rho_{\beta,\pm} \right| \le \zeta \\ \\ 0 & \text{otherwise} \end{cases} \tag{6.9}$$

and, analogously to (2.2), denoting by Y the centers of all the cubes of $\mathcal{D}^{(\ell_{-,\gamma})}$ contained in the cube of $\mathcal{D}^{(\ell_{+,\gamma})}$ labelled by C_x,

$$\theta(q_x) = \prod_{y\in Y} \mathbf{1}_{\eta^{(\zeta,\ell_{-,\gamma})}(q;y)=1} - \prod_{y\in Y} \mathbf{1}_{\eta^{(\zeta,\ell_{-,\gamma})}(q;y)=-1}$$

Peierls estimates

Given $\beta \in (\beta_c, \beta_0)$ there are $c > 0$, $\zeta > 0$, $\epsilon > 0$, $\gamma^* > 0$ and $\lambda_{\beta,\gamma}$, $\gamma \le \gamma^*$, so that the bound (2.9) holds for the Hamiltonian $H_{\gamma,\lambda_{\beta,\gamma},\beta}$ (which plays the role of the energy H_0 of Section 2) with

$$C_p = c\zeta^2 \ell^d_{-,\gamma} \tag{6.10}$$

with the contours defined using $\zeta > 0$ and $\epsilon > 0$.

Stability conditions

Denoting by $H_{\gamma,\lambda,\beta}(q|q')$ the energy of the configuration q in interaction with the configuration q', there is a constant $b = b(\lambda, \beta)$ so that

$$H_{\gamma,\lambda,\beta}(q|q') \ge b|q| \tag{6.11}$$

uniformly in γ and q'. From such a strong property, the stability estimates of Section 2 easily follow. Also the cut and paste condition (2.16) is easily seen to hold with $n = 4$.

Uniqueness regime

By applying Theorem 2.1 we deduce that for γ small enough, any DLR measure (in the LMP model all DLR measures are stable) is in $\mathcal{G}^{\text{sign of}(\lambda-\lambda_{\beta,\gamma})}$ for

$$0 < |\lambda - \lambda_{\beta,\gamma}| \le \hat{\lambda} \tag{6.12}$$

with $\hat{\lambda} = \hat{c}\gamma^{2\epsilon}$. This complements the result in [**9**] about coexistence of phases at $\lambda = \lambda_{\beta,\gamma}$.

A more accurate analysis would show that $\mathcal{G}^{\text{sign of}(\lambda-\lambda_{\beta,\gamma})}$ consists of a single element if $\lambda \ne \lambda_{\beta,\gamma}$, while at the coexistence point, each $\mathcal{G}^{\pm}$ has only one element.

An extension of the methods used in [**3**] would allow to prove that uniqueness extends to the region $|\lambda - \lambda_{\beta,\gamma}| > \hat{\lambda}$.

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Anton Bovier, Weierstrass-Institute for Applied Analysis and Stochastics, Mohrenstrasse 39, 10117 Berlin, Germany
E-mail address: `bovier@wias-berlin.de`

Immacolata Merola, Dipartimento di Matematica, Università di Roma Tor Vergata, 00133 Roma, Italy
E-mail address: `merola@mat.uniroma2.it`

Errico Presutti, Dipartimento di Matematica, Università di Roma Tor Vergata, 00133 Roma, Italy
E-mail address: `presutti@mat.uniroma2.it`

Miloš Zahradník, Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, 186 00 Prague, Sokolovská 83
E-mail address: `mzahrad@karlin.mff.cuni.cz`