

6. Übungsblatt „Mathematik III für Physiker“

Due on (28.11.2011) in the exercise class

Class exercise.

P.1 Which of the following functions can be continued analytically in z_0 ?

$$f_1(z) = \frac{z^2 + 1}{z - i}, z_0 = i \quad f_2(z) = \frac{e^z - 1}{z}, z_0 = 0, \quad f_3(z) = \frac{z}{(e^z - 1)^2}, z_0 = 0..$$

P.2 All of the following functions have a singularity in $z = 0$:

$$g_1(z) = \frac{z}{e^z - 1}, \quad g_2(z) = \sin\left(\frac{1}{z}\right), \quad g_3(z) = \frac{1}{\sin z}.$$

Which singularities are isolated? In case of isolated singularities, decide whether they are removable, a pole, or an essential singularity.

1. (Cauchy Formula)

[3 Pkt]

Here and below, let $\Gamma_r(k) : [0, 2\pi] \rightarrow \mathbb{C}$, $\theta \mapsto k + re^{i\theta}$. Compute:

- a) $\int_{\Gamma_2(0)} \frac{ze^z}{(z-1)^4} dz$
- b) $\int_{\Gamma_1(0)} \frac{\sin z}{z^4} dz$
- c) $\int_{\Gamma_2(2)} \frac{z}{z^4-1} dz$

2. (Complex integration)

[3 Pkt]

Let f be an analytic Function on $\mathbb{C} \setminus \{1, 2, 3\}$ such that

$$\int_{\Gamma_{1/2}(k)} f(z) dz = a_k, \quad (a_k \in \mathbb{C})$$

for all $k \in \{1, 2, 3\}$. Compute

$$\int_{\Gamma_4(0)} f(z) dz, \quad \int_{\Gamma_{5/2}(0)} f(z) dz, \quad \int_{\Gamma_1(5/2)} f(z) dz.$$

3. (Complex integration)

[4 Pkt]

Show that

$$I_1 := \int_0^{2\pi} e^{\cos(\theta)} \cos(\theta + \sin \theta) d\theta = 0$$

$$I_2 := \int_0^{2\pi} e^{\cos(\theta)} \sin(\theta + \sin \theta) d\theta = 0.$$

Hint: show that $I_1 + iI_2 = 0$.

4. (Complex integration)

[6 Pkt]

In this exercise we shall prove the following formula, valid for any $n \in \mathbb{N}$:

$$\binom{2n}{n} = \sum_{k=0}^n \binom{n}{k}^2.$$

a) Prove first that

$$\frac{1}{2\pi i} \int_{\Gamma_1(0)} z^* z^m (z^*)^n dz = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n. \end{cases}$$

b) Prove then that

$$\frac{1}{2\pi i} \int_{\Gamma_1(0)} z^* (z+1)^n (z^*+1)^n dz = \sum_{k=0}^n \binom{n}{k}^2$$

Hint: use the binomial theorem $(1+z)^n = \sum_{k=0}^n \binom{n}{k} z^k$ and 4.a)

c) Finally, show that

$$\begin{aligned} \frac{1}{2\pi i} \int_{\Gamma_1(0)} z^* (z+1)^n (z^*+1)^n dz &= \frac{1}{2\pi i} \int_{\Gamma_1(0)} z^* (2+z+z^*)^n dz \\ &= \frac{1}{2\pi i} \int_{\Gamma_1(0)} \frac{(z+1)^{2n}}{z^{n+1}} dz = \binom{2n}{n}. \end{aligned}$$

5. (Residues)

[4 Pkt]

Compute

$$\int_0^{2\pi} (\sin \theta)^{2n} d\theta$$

for $\theta \in \mathbb{N}$.

Hint: use that $\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$ and use the formula for residues.