

5. Übungsblatt „Mathematik III für Physiker“

Due on (21.11.2011) in the exercise class

Class exercise.

1. Let $\gamma : [0, 2\pi] \rightarrow \mathbb{C}$ mit $\gamma(t) = \frac{1}{2} + \frac{e^{it}}{4}$ und $f(z) = \frac{e^z \sin(z)}{z}$. Compute $\int_{\gamma} \frac{f(z)}{z-2} dz$.
2. Let $\gamma : [0, 2\pi] \rightarrow \mathbb{C}$ mit $\gamma(t) = \frac{1}{4} + \frac{e^{it}}{2}$ und $f(z) = \frac{(z-3)e^z}{z}$. Compute $\int_{\gamma} \frac{f(z)}{z-1} dz$.

1. (Cauchy Formula)

[4 Pkt]

Compute the following integrals :

- a) $\int_{|z+1|=1} \frac{dz}{(z+1)(z-1)^3}$
- b) $\int_{|z+2i|=3} \frac{dz}{z^2+\pi^2}$
- c) $\int_{|z|=r} \frac{dz}{(z-a)^n(z-b)^m}$ für $|a| < r < |b|$ und $n, m \geq 1$
- d) $\int_{|z|=2} \frac{1}{z^2+2z+1} dz$.

2. (Complex Integration)

[5 Pkt]

Consider the path integral $\int_{\partial R} \frac{1}{z} dz$, where R is the rectangle with vertices $r+i, -r+i, -r-i, r-i$ ($r > 0$), and compute the integral

$$\int_{-\infty}^{\infty} \frac{1}{1+t^2} dt.$$

3. (Complex Integration)

[5 Pkt]

Compute the socalled *Fresnel integrals*

$$\int_0^{\infty} \sin(x^2) dx, \quad \int_0^{\infty} \cos(x^2) dx.$$

Hint: from the Lecture we know that $\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$. Moreover, $\lim_{r \rightarrow \infty} \int_{\gamma_r} e^{-z^2} dz = 0$ for $\gamma_r : [0, \pi/4] \rightarrow \mathbb{C}, \theta \mapsto re^{i\theta}$.

4. (Complex Integration)

[6 Pkt]

Let f be analytic on $\{|\Im(z)| < 1\}$ and such that

$$\lim_{|\Re(z)| \rightarrow \infty} f(\Re(z)) = 0.$$

Assume furthermore that $\int_{-\infty}^{\infty} f(x)dx$ exists. Prove that for all $\alpha \in (-1, 1)$,

$$\int_{-\infty}^{\infty} f(x + i\alpha)dx = \int_{-\infty}^{\infty} f(x)dx.$$

Deduce from this that

$$\int_{-\infty}^{\infty} e^{-x^2} \cos(\alpha x)dx = \sqrt{\pi}e^{-\alpha^2/4}.$$