Institut für Angewandte Mathematik
Markov Processes

Prof. Dr. A. Bovier / Dr. E. Petrou

Exercise sheet 10

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Exercise 1 **(10 Points)**

Let $b_i : [0,T]\times\mathbb{R}^d \to \mathbb{R}$, $1 \le i \le d$, be uniformly bounded functions, and $f : \mathbb{R}^d \to \mathbb{R}$, $g : [0,T]\times\mathbb{R}^d \to \mathbb{R}$ continuous functions satisfying, $f(x) \ge 0$ and $g(t,x) \ge 0$ for all $x \in \mathbb{R}^d$ and $t \in [0,T]$. If $u(t,x)$ is a solution to the Cauchy problem

$$-\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + (b, \nabla u) + g, \text{ in } [0,T) \times \mathbb{R}^d,$$

$$u(T,x) = f(x), \quad x \in \mathbb{R}^d,$$

and

$$\max_{0\le t\le T} |u(t,x)| \le M e^{\mu\|x\|^2}, \quad x \in \mathbb{R}^d,$$

for some $M > 0$ and $0 < \mu < C(T,d)$, where C is a positive constant dependent on T, d, then we have the following stochastic representation

$$\begin{aligned} u(t,x) &= E^x\left[f(B_{T-t})\exp\left\{\int_0^{T-t}(b(t+\theta,B_\theta),dB_\theta) - \frac{1}{2}\int_0^{T-t}\|b(t+\theta,B_\theta)\|^2 d\theta\right\}\right] \\ &+ E^x\left[\int_0^{T-t} g(t+s,B_s)\exp\left\{\int_0^s (b(t+\theta,B_\theta),dB_\theta) - \frac{1}{2}\int_0^{T-t}\|b(t+\theta,B_\theta)\|^2 d\theta\right\} ds\right] \end{aligned}$$

where (B_t) is a d-dimensional Brownian motion.

Exercise 2 **(10 Points)**

Let X be a d-dimensional stochastic process satisfying the integral equation

$$X_s^{(t,x)} = x + \int_t^s b(X_\theta^{(t,x)})d\theta + \int_s^t \sigma(X_\theta^{(t,x)})dB_\theta, \quad t \le s < \infty,$$

where (B_t) is a d-dimensional Brownian motion, $b_i, \sigma_{ij} : \mathbb{R}^d \to$ are continuous functions satisfying the linear growth condition

$$\|b(x)\|^2 + \|\sigma(x)\|^2 \le K^2(1 + \|x\|^2), \text{ for every } x \in \mathbb{R}^d,$$

where $K > 0$ is a constant. Assume that for every open, bounded domain $D \subset \mathbb{R}^d$, for some $1 \le l \le d$, $\min_{x\in\overline{D}} a_{ll}(x) > 0$ holds true. Furthermore, suppose that there exists a function $f : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ in C^2, which satisfies

$$\mathcal{L}f(x) \le 0 \text{ on } \mathbb{R}^d \setminus \{0\}$$

and is such that $F(r) \equiv \min_{\|x\|=r} f(x)$ is strictly increasing with $\lim_{r\to\infty} F(r) = \infty$.

(a) Prove that we have the *recurrence property*

$$P^x(\tau_r < \infty) = 1, \forall x \in \mathbb{R}^d \setminus \overline{D}_r, \tag{1}$$

for every $r > 0$, where $D_r = \{x \in \mathbb{R}^d : \|x\| < r\}$ and $\tau_r = \inf\{t \ge 0 : X_t \in \overline{D}_r\}$.

(b) Show that (1) holds in the case

$$(x, b(x)) + \frac{1}{2}\sum_{i=1}^{d} a_{ii}(x) \leq \frac{(x, a(x)x)}{\|x\|^2}, \forall x \in \mathbb{R}^d \setminus \{0\}.$$

(c) Show that if

$$\mathcal{L}f(x) \leq -1 \text{ on } \mathbb{R}^d \setminus \{0\},$$

then we have the *positive recurrence property*

$$E^x[\tau_r] < \infty, \forall x \in \mathbb{R}^d \setminus \overline{B}_r.$$

Gesamt: 20 Punkte