**Institut für Angewandte Mathematik**
**Markov Processes**

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# Exercise sheet 8

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## Exercise 1

**(10 Points)**

Consider the operators

$$\mathcal{D}(G_1) = \{f \in C^2([0,\infty)) \,:\, f''(0) = 0\}, \qquad G_1 f = \frac{1}{2} f'' \,,$$
$$\mathcal{D}(G_2) = \{f \in C^2([0,\infty)) \,:\, f'(0) = 0\}, \qquad G_2 f = \frac{1}{2} f'' \,,$$

corresponding to absorbing and reflecting Brownian motion respectively (see Sh.6 ex. 1 and Sh.7 ex. 2). Let $g \in C^2(\mathbb{R})$ be an odd function.

a) Show that the martingale problems associated to $G_1$ and $G_2$ are dual with respect to $(f, 0, 0)$, where $f(x, y) = g(x + y) + g(x - y)$.

b) Let $X$ be absorbing Brownian motion and let $Y$ be reflecting Brownian motion. Use the result in part a) to show that

$$\mathbb{P}(X(t) > y \,|\, X(0) = x) = \mathbb{P}(Y(t) < x \,|\, Y(0) = y) \,.$$

## Exercise 2

**(10 Points)**

Let $X$ be a one-dimensional Lévy process with characteristic exponent $\psi$ and characteristics $(b, \sigma^2, \nu)$. Let $\{P_t\}_{t\geq 0}$ be the associated Feller-Dynkin semigroup and $G$ be its infinitesimal generator. Prove that:

(a) For each $t \geq 0$, $f \in C_c^\infty(\mathbb{R})$[1], $x \in \mathbb{R}$,

$$(P_t f)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{iux} e^{-t\psi(u)} \hat{f}(u) du,$$

where $\hat{f}$ is the Fourier transform of $f$.

(b) For each $f \in C_c^\infty(\mathbb{R})$, $x \in \mathbb{R}$,

$$(Gf)(x) = -\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{iux} \psi(u) \hat{f}(u) du.$$

(c) For each $C_c^\infty(\mathbb{R})$, $x \in \mathbb{R}$,

$$(Gf)(x) = bf'(x) + \frac{1}{2}\sigma^2 f''(x) + \int_{\mathbb{R}\setminus\{0\}} \left[ f(x+y) - f(x) - yf'(x)\mathbf{1}_{|y|<1} \right] \nu(dy).$$

**Hint:** Use the fact that there exists $C > 0$ such that

$$|\psi(u)| \leq C(1 + u^2), \text{ for all } u \in \mathbb{R}.$$

Gesamt: 20 Punkte

[1] $C_c^\infty(\mathbb{R})$ denotes the set of smooth functions with compact support.