Institut für Angewandte Mathematik
Markov Processes

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Exercise sheet 7

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Exercise 1 **(10 Points)**

Let $c : \mathbb{R} \to \mathbb{R}$ be strictly positive and continuous on $\mathbb{R}$, and consider a Feller-Dynkin process whose generator, when restricted to C^2 functions f with compact support, is given by

$$\mathcal{L}f(x) = \frac{1}{2}c(x)f''(x).$$

(a) Show that if $a < x < b$ and τ is the hitting time of $\{a, b\}$, then

$$E^x\tau = \int_a^b \frac{2}{c(z)} \frac{(x \wedge z - a)(b - x \vee z)}{b - a} dz.$$

Hint: Find an appropriate martingale associated to the Feller-Dynkin process and apply the strong Markov property.

(b) Use part (a) to show that if τ is the hitting time of a, then for $x > a$, $E^x\tau < \infty$ if and only if $\int_0^\infty 1/c(x)dx < \infty$.

Exercise 2 (Killed Brownian motion) **(10 Points)**

Let $\{B_t\}_{t\in\mathbb{R}_+}$ be a Brownian motion and $T = \inf\{t \geq 0 : B_t = 0\}$. Furthermore, let X be a process defined by

$$X_t = B_t \text{ on } \{t < T\} \text{ and } X_t = \partial \text{ on } \{t \geq T\}.$$

This process is called a Brownian motion *killed* (absorbed) at 0. Assume that initial distribution of X is $P_0(x, \cdot) = \delta_x(\cdot)$, where δ_x denotes the Dirac measure for all $x \in \mathbb{R}_+$.

(a) Prove that X is a Markov process on $(0, \infty)$ with transition function given by the density

$$\frac{1}{\sqrt{2\pi t}} \left[\exp\left(-\frac{1}{2t}(y - x)^2\right) - \exp\left(-\frac{1}{2t}(y + x)^2\right)\right], \text{ for } x > 0 \text{ and } y > 0.$$

Hint: In order to find the transition function use the reflection principle.

(b) Is the transition function honest?

(c) Is the Brownian motion absorbed at 0, i.e. $X_t = B_{t\wedge T}$, a Markov process? If so, find the transition function.

Gesamt: 20 Punkte