Institut für Angewandte Mathematik
Markov Processes

Prof. Dr. A. Bovier / Dr. E. Petrou

Exercise sheet 6

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Exercise 1 **(6 Points)**

Let $\{B_t\}_{t\in\mathbb{R}_+}$ be a Brownian motion and $X_t = |B_t|$ for $t \geq 0$. Assume that initial distribution of X is $P_0(x,\cdot) = \delta_x(\cdot)$, where δ_x denotes the Dirac measure for all $x \in \mathbb{R}_+$.

(a) Prove that X is a Markov process with transition function given by the density

$$\frac{1}{\sqrt{2\pi t}}\left[\exp\left(-\frac{1}{2t}(y-x)^2\right) + \exp\left(-\frac{1}{2t}(y+x)^2\right)\right]\mathbf{1}_{\{y\geq 0\}}.$$

(b) Prove that the associated semigroup P_t is a Feller-Dynking semigroup.

Exercise 2 **(2 Points)**

Let $\{P_t\}_{t\in\mathbb{R}_+}$ be an honest Feller-Dynkin semigroup. Prove, without using the Hille-Yosida Theorem, that the associated generator G is dissipative.

Exercise 3 **(6 Points)**

Let P_t be a the semigroup associated with a Feller-Dynkin Markov process $\{X_t\}_{t\in\mathbb{R}_+}$, with $X_0 = x$, a.s. $x \in \mathbb{R}$. Furthermore, let T be an a.s. finite $\mathcal{F}_{t^+}$-stopping time. Prove that:

(a) The process $Y_t = X_{T+t}$ is a Markov process with respect to $\mathcal{F}_{T+t}$.

(b) If X has stationary independent increments, the process $\{X_{T+t} - X_T\}_{t\in\mathbb{R}_+}$ is independent of $\mathcal{F}_T$ and it has the same law as X.

Exercise 4 **(6 Points)**

Let $(\tau_a, a \in \mathbb{R})$ be the translation group acting in $B(\mathbb{R})$, such that $(\tau_a f)(x) = f(x-a)$ for each $a, x \in \mathbb{R}$,$f \in B(\mathbb{R})$. Then a semigroup $\{P_t\}_{t\in\mathbb{R}_+}$ is called *translation invariant* if

$$P_t\tau_a = \tau_a P_t,$$

for all $t \in \mathbb{R}_+, a \in \mathbb{R}$.

Prove that a Feller-Dynkin semigroup $\{P_t\}_{t\in\mathbb{R}_+}$, associated with a process $\{X_t\}_{t\in\mathbb{R}_+}$ for which $X_0 = 0$ a.s., is translation invariant if and only if X is a Lévy process.

Hint: Use the following result.
Theorem. Let $\{q_t\}_{t\in\mathbb{R}_+}$ be a family of probability measures, with $q_0 = \delta_0$ (Dirac measure at 0), such that $q_{s+t} = q_s * q_t$ for all $s, t \geq 0$, and $\lim_{t\downarrow 0}\int_{\mathbb{R}} f(x)q_t(dx) = f(0)$ for all $f \in C_0(\mathbb{R})$. Then there exists a Lévy process X, such that the law of X_t is q_t.

Gesamt: 20 Punkte